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CONTINUITY PROPERTIES OF DERIVATIVES OF SEQUENCES OF FUNCTIONS[1]

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In a recent paper Dvoretzky [1] discusses an interesting generalization of a theorem of Walsh [2]. A striking supplement to Dvoretzky's theorem is the following one.

THEOREM. *There exists a sequence of functions*

$$\{f_n\}_1^\infty, \quad f_n \in C^1(-\infty, \infty),$$

with $\lim_{n\to\infty} f_n(x) \equiv 0$, *such that: if* N_1 *is any subsequence of the natural numbers with the property that there exists a sequence* x_{n_1}, $n_1 \in N_1$, *satisfying*

$$f'_{n_1}(x_{n_1}) = 0, \quad \text{and} \quad \lim_{n_1\to\infty} x_{n_1} = 0, \tag{1}$$

then the sequence N_2 *complementary to* N_1 *(i.e.,* N_2 *contains exactly those natural numbers omitted by* N_1*) is infinite and*

$$\limsup_{n_2\to\infty} \int_0^h |f'_{n_2}(x)|\, dx = \infty \tag{2}$$

for every $h>0$.

PROOF. Let $\epsilon_n \downarrow 0$ and let $\{\lambda_n\}$ be the sequence 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, $\cdots$. Let $\{\mu_n\}$ be a sequence of positive numbers such that

$$\epsilon_n\mu_n/\lambda_n \to \infty, \qquad n \to \infty. \tag{3}$$

The functions $f_n(x)$ shall be odd and

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$$f_n(x) = \begin{cases} \epsilon_n \sin \lambda_n x, & |x| \leqq \pi/2\lambda_n, \\ \epsilon_n \cos \mu_n \left(x - \dfrac{\pi}{2\lambda_n} \right), & x > \pi/2\lambda_n. \end{cases}$$

It is easily verified that $f_n \in C^1$ and $f_n(x) \to 0$. The smallest zeros of $f_n'(x)$ are $\pm\pi/(2\lambda_n)$. Because of (1) and the nature of λ_n, any possible N_2 will contain infinitely many n_2 associated with each possible value of λ. Thus N_2 splits into disjoint infinite sequences M_p, $p \geqq 1$, such that

$$\lambda_{m_p} = p, \qquad m_p \in M_p.$$

For a given $h>0$, choose p such that $\pi/p \leqq h$. Then

$$\begin{aligned} \int_0^h | f'_{m_p}(x) | \, dx &> \int_{\pi/(2p)}^{\pi/p} | f'_{m_p}(x) | \, dx \\ &= \epsilon_{m_p} \mu_{m_p} \int_{\pi/(2p)}^{\pi/p} \left| \sin \mu_{m_p} \left(x - \frac{\pi}{2p} \right) \right| dx \\ &= \epsilon_{m_p} \int_0^{\pi \mu_{m_p}/(2p)} | \sin t | \, dt \geqq \epsilon_{m_p} [\mu_m / p], \end{aligned}$$

and (2) follows from this inequality and (3).

Finally we note that Dvoretzky's use of the word "clearly" in the third line after equation (6) is dubious.

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