BLOCH FUNCTIONS OF THE THIRD KIND AND THE CONSTANT $\mathfrak{A}$

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Let S be the class of functions $w=f(z)=z+a_2z^2+\dots$ analytic and schlicht for $|z|<1$. For the purpose of obtaining a lower bound on $\mathfrak{A}$ (defined below) Landau [1] considered a subclass $T\subset S$ for which $|f'(z)|(1-|z|^2)\leq 1$. He showed that $a_2=0$, $|a_3|\leq 1/3$. He also proved the inequality $\mathfrak{A}>.566$. Quite recently E. Reich [2] proved $\mathfrak{A}>.569$, which is very slightly better than Landau's original bound. The purpose of this paper is to give a better bound for $\mathfrak{A}$ and $|a_3|$.

Before proceeding it is convenient to recall the definition of $\mathfrak{A}$ and the definition of a Bloch function of the third kind [3] which belongs to T .

DEFINITION. Let A_f denote the least upper bound of the radii of all circles contained in the map of $w=f(z)$ which is schlicht for $|z|<1$ and has $|f'(0)|=1$. Then $\mathfrak{A}=\min A_f$ is the Landau constant.

DEFINITION. If the map of $w=f(z)$ contains no circle of radius greater than $\mathfrak{A}$, then f is called a Bloch function of the third kind.

In the following we shall use the abbreviation B.F. for a Bloch function of the third kind.

THEOREM 1. $\mathfrak{A}>.629$.

PROOF. Let $\omega=\zeta+a_3\zeta^3+\dots$ be a B.F. then $w=f(z)=z+|a_3|z^3+\dots$ is also a B.F. But

$$|f(z)| \leq \frac{1}{2} \log \frac{1+|z|}{1-|z|} = |z| M(|z|).$$

For arbitrary fixed s , $0<s<1$, put $f(z, s)=f(sz)/s$. Then

$$|f(z, s)| \leq M(s), \quad |z| < 1.$$

Now write

$$\tilde{f}(z, s, t) = tM(s)(\Phi((f(z, s)/tM(s))^3))^{1/3} = z + |a_3|s^2z^3 + \dots,$$

where $\Phi(z')=z'/(1\pm tz')^2$, $t>0$, $|z'|<1/t$. If $f(z)$ omits c , then $\tilde{f}(z, s, t)$ omits

$$\gamma(s) = (c/s)(1 \pm t^{-2}c^3/s^3M^3(s))^{-2/3}.$$

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¹ Obtained by rotating the ω and ζ -plane through an angle equal to $-2^{-1} \arg a_3$.

If $-\pi/2 \leq \arg c^3 \leq \pi/2$, we take the sign $+$. For a suitable choice of t , $\gamma(s)$ is real, >0 and $s\gamma(s) \leq |c|$. In the alternative case, we take the sign $-$ and replace c^3 by $-c^3$. Again for a suitable choice of t , $\gamma(s)$ is real and (<0) and $s|\gamma(s)| \leq |c|$. Thus in all cases, for a proper choice of t ,

$$F(z, s) = \frac{\gamma(s) \bar{f}(z, s, t)}{\gamma(s) - \bar{f}(z, s, t)} = z + b_2 z^2 + b_3 z^3 + \dots,$$

where $b_2 = 1/\gamma(s)$, $b_3 = 1/\gamma^2(s) + |a_3|s^2$, is schlicht for $|z| < 1$ and omits $\gamma(s)$ real (≤ 0 according to the cases).

Now, since $F(z, s)$ is analytic and schlicht for $|z| < 1$, then [4; 5]

$$|b_3 - \alpha b_2^2| \leq 2e^{-2\alpha/(1-\alpha)} + 1, \quad \text{for all } 0 < \alpha < 1.$$

I.e.,

$$(1) \quad |a_3|s^2 + (1 - \alpha)/\gamma^2(s) \leq 2e^{-2\alpha/(1-\alpha)} + 1.$$

Hence

$$(1 - \alpha)/\gamma^2(s) \leq 2e^{-2\alpha/(1-\alpha)} + 1, \quad \text{for all } 0 < \alpha < 1.$$

In particular for $\alpha = .45$, a numerical computation yields

$$|\gamma(s)| > .629.$$

If c is nearest to the origin then $\Re \geq |c|$. Hence $.629s < s|\gamma(s)| \leq |c| \leq \Re$, i.e., $.629s < 21$. Making $s \rightarrow 1$ we have the result.

THEOREM 2. *If f is a B.F. and belongs to T , then $|a_3| < 1/9$.*

PROOF. (1) and $s|\gamma(s)| \leq \Re$ imply

$$s^2(|a_3| + (1 - \alpha)\Re^{-2}) \leq 2e^{-2\alpha/(1-\alpha)} + 1,$$

and in particular for $s \rightarrow 1$

$$|a_3| + (1 - \alpha)\Re^{-2} \leq 2e^{-2\alpha/(1-\alpha)} + 1.$$

In particular for $\alpha = 1/2$, $\Re < .6565$ [6], and a numerical computation yields the result stated in the theorem.

It is very likely that $a_3 = 0$ and that the map of the corresponding B.F. is three-fold symmetric.

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A T -SYSTEM WHICH IS NOT A BERNSTEIN SYSTEM¹

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(I) Let E be a separable real Banach space, let $\{u_i\}_1^\gamma$ be a fundamental² linearly independent sequence (finite or infinite) of points of E , and for each $0 \leq n < \gamma + 1$, let L_n be the linear extension of $\{u_i\}_1^n$, with $L_0 = \{\phi\}$. Then, $\{u_i\}_1^\gamma$ is called a T -system for E if for each x in E and for each $0 \leq n < \gamma + 1$ there is a unique $P_n(x) = \sum_{i=1}^n x_{i,n} u_i$ in L_n such that $\|x - P_n(x)\| = \inf_{Q \in L_n} \|x - Q\|$. The deviation sequence, $\{\delta_i(x)\}_0^\gamma$, of x relative to $\{u_i\}_1^\gamma$ is defined by the conditions: (1) $|\delta_i(x)| = \|x - P_i(x)\|$ for all $0 \leq i < \gamma + 1$ and (2) for $0 \leq i < \gamma + 1$, if $\delta_i(x) \neq 0$, then $\text{sgn } \delta_i(x) = \text{sgn } x_{i',i'}$, where i' is the least integer greater than i for which $x_{i',i'} \neq 0$. If $\{\delta_i(x)\}_0^\gamma = \{\delta_i(y)\}_0^\gamma$ implies that $x = y$, then the T -system $\{u_i\}_1^\gamma$ is called a Bernstein system for E .

The definitions of T -systems and Bernstein systems were essentially introduced by Kadeč in [4], in generalizing some results of Bernstein [2]. The proofs in [2] and [4] show or can be used to show that any two separable Banach spaces are homeomorphic if they possess Bernstein systems of the same cardinality and that for finite dimensional spaces every T -system is a Bernstein system. Now, given a separable Banach space we can always find a fundamental linearly independent sequence of points in it and by strictly convexifying the space [3] such a sequence becomes a T -system. In [4], Kadeč states as an open question: Is every T -system a Bernstein system? Since every separable infinite dimensional Banach space has a T -system under some equivalent norm, an affirmative answer to this would have proved that all such spaces are homeomorphic.

It is the purpose of this paper to exhibit a T -system which is not a Bernstein system. In (II), we describe such a system for the space c_0 under a norm equivalent to the usual sup norm.

We note that Kadeč [4] showed that c_0 does possess a Bernstein

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² Most terms not defined in the text may be found in Banach [1].