BOUNDED GROUPS OF TRANSFORMATIONS

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This note is intended as a comment on a result of Bochner and Martin's and their notation will be used.¹

Formal power series mapping an n -dimensional complex space into itself are considered. A typical inner transformation T mapping $x = (x_1, \dots, x_k)$ into $y = (y_1, \dots, y_k)$ is given by

$$y_j = C_1^j x_1 + \dots + C_k^j x_k + \sum_{n_1 + \dots + n_k > 1} C_{n_1 \dots n_k}^j x_1^{n_1} \dots x_k^{n_k}, \quad j = 1, \dots, k.$$

The linear part $L(T)$ of T is the transformation

$$y_j = C_1^j x_1 + \dots + C_k^j x_k.$$

Transformations with determinants $|C_i^j|$ nonzero form a group where the group operation is defined in the obvious way in terms of successive substitutions.

A group of transformations $T(\alpha)$ depending on a parameter α is said to be bounded if

$$|C_{n_1 \dots n_k}^j(\alpha)| \leq K_{n_1 \dots n_k}^j < \infty$$

for all α . Bochner and Martin prove that a coordinate system exists in which the elements of a bounded group take particularly simple form.

Their theorem is:

Any bounded group of inner transformations $T(\alpha)$ is similar to the group of its linear parts $L(T(\alpha))$: there exists a transformation S whose linear part is the identity

$$S: x_j' = x_j + (\text{higher powers}), \quad j = 1, \dots, k$$

such that for all α

$$T(\alpha) = S^{-1}L(T(\alpha))S.$$

The question of the uniqueness of S is not considered and this point is our concern here.

Now suppose that a bounded group of substitutions contains a

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¹ S. Bochner and W. T. Martin, *Several complex variables*, Princeton, 1948, p. 19, Theorem 8.

subgroup of linear transformations and let A be any member of the subgroup.

Since $\dots, A^{-n}, \dots, A^{-1}, A^0, A, \dots, A^n, \dots$ is a bounded sequence of transformations

$$(1) \quad A = H M H^{-1},$$

where M is the multiplication

$$(2) \quad y_j = e^{i2\pi\alpha_j} x_j, \quad \alpha_1, \dots, \alpha_k \text{ real,}$$

and H is linear.

Now from the above theorem $A = S^{-1} A S$ i.e.

$$(3) \quad S A = A S.$$

Substituting from (1)

$$H^{-1} S H M = M H^{-1} S H.$$

Writing

$$(4) \quad H^{-1} S H = V$$

this becomes

$$(5) \quad V M = M V,$$

where V is of the form

$$(6) \quad y_j = x_j + \sum_{n_1 + \dots + n_k > 1} a_{n_1 \dots n_k}^j x_1^{n_1} \dots x_k^{n_k}.$$

Substituting from (2), $M V$ is the substitution

$$(7) \quad y_j = e^{i2\pi\alpha_j} x_j + \sum_{n_1 + \dots + n_k > 1} a_{n_1 \dots n_k}^j e^{i2\pi\alpha_j n_1} x_1^{n_1} \dots x_k^{n_k}$$

and $V M$ the substitution

$$(8) \quad y_j = e^{i2\pi\alpha_j} x_j + \sum_{n_1 + \dots + n_k > 1} a_{n_1 \dots n_k}^j e^{i2\pi(n_1\alpha_1 + \dots + n_k\alpha_k)} x_1^{n_1} \dots x_k^{n_k}.$$

Now (7) and (8) are identical if

$$a_{n_1 \dots n_k}^j e^{i2\pi\alpha_j} = a_{n_1 \dots n_k}^j e^{i2\pi(n_1\alpha_1 + \dots + n_k\alpha_k)}$$

i.e.

$$(9) \quad a_{n_1 \dots n_k}^j = 0 \text{ if } n_1\alpha_1 + \dots + n_k\alpha_k \not\equiv \alpha_j \pmod{1}.$$

For most sets of $\alpha_1, \dots, \alpha_k$, the equation in $n_1, \dots, n_k$

$$(10) \quad n_1\alpha_1 + \cdots + n_k\alpha_k = \alpha_j \pmod{1}$$

will have no solution e.g. if $\alpha_1 = e, \alpha_2 = e^2, \dots, \alpha_k = e^k$ (10) has no solution since e is not algebraic. In such a case V is the identity, consequently S is also the identity and the subgroup of linear transformations is, in fact, the complete group. However for special sets of $\alpha_1, \dots, \alpha_k$, V need not be linear.

The above work gives information on the uniqueness of S in the general case. If

$$T = S_1^{-1}L(T)S_1$$

and

$$T = S_2^{-1}L(T)S_2$$

then on writing $S_2 = VS_1$ we find

$$(11) \quad L(T) = V^{-1}L(T)V.$$

If this last equation implies that the substitution V (whose linear part is the identity) is the identity, then $S_1 = S_2$ and S is unique.

Here we are considering only a single transformation of the group and obviously to determine all restrictions on the arbitrariness of S it may be necessary to consider the set of all elements of the group.

EXAMPLE. Let $\alpha_1 = 0, \alpha_2 = 1/2, H = I$.

Then (9) becomes

$$\alpha_{n_1 n_2}^1 = 0 \text{ if } \frac{1}{2} n_2 \not\equiv 0 \pmod{1}, \quad \alpha_{n_1 n_2}^2 = 0 \text{ if } \frac{1}{2} n_2 \not\equiv \frac{1}{2} \pmod{1}.$$

$S = V$ can be any transformation of the form

$$\begin{aligned} y_1 &= x_1 + \sum_{n_1=0,1,2,\dots} \sum_{n_2=0,2,4,\dots} a_{n_1 n_2}^1 x_1^{n_1} x_2^{n_2}, \\ y_2 &= x_2 + \sum_{n_1=0,1,2,\dots} \sum_{n_2=1,3,\dots} a_{n_1 n_2}^2 x_1^{n_1} x_2^{n_2}. \end{aligned}$$

If the linear subgroup contains not only the substitution

$$y_1 = x_1, \quad y_2 = -x_2$$

but also the substitution

$$y_1 = -x_1, \quad y_2 = x_2$$

then S must be the identity.