# DIFFERENTIAL EQUATIONS INVOLVING A PARAMETRIC FUNCTION[1]

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**1. Introduction.** It is the purpose of this paper[2] to find conditions on the function $f(t, u)$ under which the differential system

$$\begin{cases} \dfrac{dz}{dt} + f[t, y(t) + z(t)] = 0, & t \in I. \\ z(0) = 0 \end{cases} \tag{1.1}$$

has a solution $z(t)$ on the unit interval $I$ for almost all choices of the function $y$ in the space $C$. Here $C$ denotes the space of functions which are continuous on the interval $I$: $0 \leqq t \leqq 1$ and which vanish at $t=0$; and "almost all" means all except a set of Wiener measure[3] zero. Under the transformation

$$z(t) = x(t) - y(t) \tag{1.2}$$

the system (1.1) goes into the equivalent nonlinear integral equation

$$y(t) = x(t) + \int_0^t f[s, x(s)]ds, \qquad t \in I, \tag{1.3}$$

so that we are seeking conditions on $f$ which make (1.3) have a solution $x \in C$ for almost every choice of $y$ in $C$.

The simplest conditions of this type which we have found, and which do not force (1.3) to have a solution for *every* $y \in C$, are given in the following theorem.

THEOREM 1. *Let $f(t, u)$ have continuous first partial derivations $f_t$ and $f_u$ in the strip $R$: $0 \leqq t \leqq 1$, $-\infty < u < \infty$, and let $f$ satisfy the three order of growth conditions*

$$f(t, u) \operatorname{sgn} u \geqq -A_1 \exp(Bu^2) \qquad \textit{in } R, \tag{1.4}$$

$$f_u(t, u) + 4g_t(t, u) \leqq 2\alpha^2u^2 + A_2 \qquad \textit{in } R, \tag{1.5}$$

Presented to the Society April 20, 1957; received by the editors February 15, 1957.

[1] This research was supported by the United States Air Force, through the office of Scientific Research of the Air Research and Development Command, under contract No. AF 18(603)-30. Reproduction in whole or in part is permitted for any purpose of the United States Government.

[2] The author wishes to thank Mr. James Yeh for carefully checking the manuscript of this paper.

[3] See, for instance, [4].

$$g(1, u) \geqq -\frac{1}{2}\alpha u^2 \cot \beta - A_3 \quad \textit{for all real } u, \tag{1.6}$$

*where*

$$g(t, u) = \int_0^u f(t, v)dv \qquad \textit{in } R, \tag{1.7}$$

*and $A_1$, $A_2$, $A_3$, $B$, $\alpha$, $\beta$ are positive constants with $\alpha<\beta<\pi$ and $B<1$. Then it follows that corresponding to almost every choice of $y\in C$, the system* (1.1) *has a solution $z\in C$, (and of course, this is the only solution defined on the interval $I$).*

The fact that this can apply in cases where classical theorems (i.e., theorems giving conditions under which there are solutions for all choices of $y$ in $C$) do not apply is shown by the counter-example

$$\begin{aligned} f(t, u) &= \frac{1}{2} u^{1/3} \sin (u^{4/3}) + \frac{1}{2} u^{5/3} \cos (u^{4/3}) \\ &= \frac{3}{8}\frac{d}{du}\left[u^{4/3} \sin u^{4/3}\right] \qquad \text{in } R. \end{aligned} \tag{1.8}$$

It is easy to see that (1.8) satisfies the conditions of the theorem, and hence that there are solutions of (1.1) for almost all $y$ in $C$ when $f$ is given by (1.8). On the other hand, we shall show in §3 that there exists at least one $y$ in $C$ for which (1.1) has no solution. Thus Theorem 1 cannot be contained in any classical theorem.

2. **Proof of Theorem 1.** Assume that the hypotheses of the theorem are satisfied, choose $\gamma=B^{-1}-1$, and let

$$\phi(t, u) = (t+\gamma)^{-1/2} \exp\left\{u^2(t+\gamma)^{-1}\right\} \qquad \text{in } R. \tag{2.0}$$

Let

$$G(t, u, \lambda) = g(t, u) + \lambda\phi(t, u), \qquad (t, u) \in R, \lambda \geqq 0, \tag{2.1}$$

so that

$$G_u(t, u, \lambda) = f(t, u) + 2\lambda(t+\gamma)^{-1}u\phi(t, u), \qquad (t, u) \in R, \lambda \geqq 0 \tag{2.2}$$

and

$$G_{u,u} + 4G_t = f_u + 4g_t, \qquad (t, u) \in R, \lambda \geqq 0. \tag{2.3}$$

From (2.1), (2.0), (1.4) we have

$$G_u \operatorname{sgn} u \geqq A_1 \tag{2.4}$$

when $|u| \geqq \lambda^{-1}B^{-3/2}A_1$ and $t \in I$ and $\lambda > 0$, and since for fixed positive $\lambda$, $G_u$ is bounded for $|u| \leqq \lambda^{-1}B^{-3/2}A_1$, $t \in I$, it follows that there is a positive function of $\lambda$ alone, $A(\lambda)$, such that

$$G_u(t, u, \lambda) \operatorname{sgn} u \geqq -A(\lambda), \qquad (t, u) \in R, \lambda > 0. \tag{2.5}$$

Now (2.5) implies that for fixed positive $\lambda$, the integral equation

$$y(t) = x(t) + \int_0^t G_u(s, x(s), \lambda)ds \tag{2.6}$$

has a solution $x \in C$ for each $y \in C$. For the equation clearly has a solution on some interval to the right of zero, and if this interval does not include $t=1$, it must be open at the right hand end and the solution must become unbounded in the neighborhood of this point. But by (2.5), this would imply that $y$ would vary by unbounded amounts as $x$ did so, contrary to the assumption that $y$ is continuous.

Since for fixed positive $\lambda$, (2.6) has a solution for each $y$ in $C$, it follows from Theorem 3 of [**3**], (using Footnote 9), that

$$\int_C \exp\{J(x, \lambda)\} d_w x = 1, \qquad \lambda > 0, \tag{2.7}$$

where

$$J(x, \lambda) = \int_0^1 K(s, x(s), \lambda)ds - 2G(1, x(1), \lambda) \quad \text{for } x \in C, \lambda \geqq 0, \tag{2.8}$$

and

$$K(t, u, \lambda) = \frac{1}{2} G_{u,u} - G_u^2 + 2G_t \qquad (t, u) \in R, \lambda \geqq 0. \tag{2.9}$$

But by (2.9), (2.2), (2.3),

$$\lim_{\lambda \to 0^+} K(t, u, \lambda) = K(t, u, 0) = \frac{1}{2} f_u + 2g_t - f^2 \qquad \text{in } R, \tag{2.10}$$

and for each fixed $x$ in $C$, we have from (2.8), (2.10), (2.1), (2.2), (2.3), by bounded convergence

$$\begin{aligned} \lim_{\lambda \to 0^+} J(x, \lambda) &= J(x, 0) \\ &= \int_0^1 K(s, x(s), 0)ds - 2g(1, x(1)). \end{aligned} \tag{2.11}$$

Moreover, it follows from (2.8), (2.9), (2.3), (2.1), (1.5), (1.6), that

$$
\begin{aligned}
J(x, \lambda) &\leqq \frac{1}{2}\int_0^1 \{f_u(s, x(s)) + 4g_t(s, x(s))\}ds - 2g(1, x(1)) \\
&\leqq \log Q(x) + \frac{1}{2} A_2 + 2A_3 \qquad \text{for } x \in C, \lambda > 0
\end{aligned} \tag{2.12}
$$

where for $x\in C$,

$$
Q(x) = \exp\left\{\alpha^2\int_0^1 [x(s)]^2 ds + \alpha[x(1)]^2 \cot\beta\right\}. \tag{2.13}
$$

Finally, we show that $Q(x)$ is integrable over $C$, by applying the transformation

$$
y(t) = x(t) - \alpha\int_0^t \cot[\alpha s + \beta - \alpha]x(s)ds, \qquad t \in I
$$

to the Wiener integral of unity, using Theorem A of [2]. We obtain (using (2.13))

$$
1 = \int_C 1 d_w y = \exp\left\{-\frac{1}{2}\alpha\int_0^1 \cot[\alpha s + \beta - \alpha]ds\right\}\int_C Q(x)d_w x,
$$

so that the integrability of $Q$ is established.

Now we take limits in (2.7) as $\lambda \to 0^+$, using (2.11), (2.12) and dominated convergence, and thus establish that (2.7) holds even when $\lambda = 0$. Hence it follows from Theorem 3 of [3], (using Footnote 9) and from (2.8), (2.9), (2.1), (2.2), (2.3), that the integral equation (1.3) has solutions $x\in C$ for almost all $y \in C$. But (1.3) is equivalent to (1.1) by the transformation (1.2), and the theorem is proved.

3. **A counterexample.** We shall now show that when $f(t, u)$ is given by (1.8), there exists a function $y\in C$ such that (1.3) (and hence also (1.1)), has no solution in $C$. We begin by constructing a certain function $x$ which does not belong to $C$ because it becomes infinite as we approach $t=1$.

Let

$$
t_n = 1 - n^{-1/3} \quad \text{and} \quad u_n = (2n\pi)^{3/4}, \qquad n = 1, 2, 3, \cdots, \tag{3.1}
$$

let $M$ be the set of monotonically increasing functions defined on $[0, 1)$ which satisfy

$$
x(0) = 0, \qquad x(t_n) = u_n, \qquad n = 1, 2, \cdots, \tag{3.2}
$$

and let $M_c$ be the subset of $M$ consisting of those elements of $M$ which are continuous on $[0, 1)$. Define the functionals $Q_n(x)$ by

$$Q_n(x) = \int_{t_n}^{t_{n+1}} f[x(s)]ds, \qquad x \in M, \tag{3.3}$$

where $f(u) \equiv f(t, u)$ is given by (1.8). We shall show the existence of an element $x$ of $M_c$ for which

$$Q_n(x) = u_n - u_{n+1} \tag{3.4}$$

for sufficiently large $n$; i.e., for which

$$x(t_{n+1}) - x(t_n) + Q_n(x) = 0 \tag{3.5}$$

for sufficiently large $n$.

To show that there is an element of $M_c$ for which (3.4) holds for all sufficiently large $n$, consider a particular interval $[t_n, t_{n+1}]$ and an element $x_1$ of $M$ which is constant on $[t_n, t_{n+1})$, so that

$$x_1(t) = u_n, \qquad t_n \leqq t < t_{n+1}.$$

Then we have by (3.3), (3.1), (1.8),

$$Q_n(x_1) = f(u_n)[t_{n+1} - t_n] = \frac{1}{2} u_n^{5/3}(t_{n+1} - t_n) > 0 > u_n - u_{n+1}.$$

Now it is clear that $M_c$ is dense in $M$ in the $L_1[t_n, t_{n+1}]$ topology, and also that $Q_n(x)$ is continuous in the $L_1[t_n, t_{n+1}]$ topology applied to the space $M$, since $u_n \leqq x(t) \leqq u_{n+1}$ when $x \in M$ and $t \in [t_n, t_{n+1}]$. Hence there is an element $x_2 \in M_c$ for which $Q_n(x_2)$ differs as little as we please from $Q_n(x_1)$, and in particular

$$Q_n(x_2) > u_n - u_{n+1}. \tag{3.6}$$

To obtain an $x$ where the inequality goes the other way, we now set

$$x_3 = u_n' \equiv [(2n + 1)\pi]^{3/4}, \qquad t_n < t < t_{n+1},$$

so that we have by (3.3), (3.1), (1.8),

$$\begin{aligned} Q_n(x_3) &= f(u_n')[t_{n+1} - t_n] \\ &= -\frac{1}{2}[(2n + 1)\pi]^{5/4}[n^{-1/3} - (n + 1)^{-1/3}], \\ &\leqq -\frac{1}{6}[2n\pi]^{5/4}[n + 1]^{-4/3}, \end{aligned} \tag{3.7}$$

while

$$u_n - u_{n+1} = (2n\pi)^{3/4} - [2(n + 1)\pi]^{3/4} \geqq -\frac{3}{4}(2\pi)^{3/4}n^{-1/4}. \tag{3.8}$$

It is clear that there exists a positive integer $N$ such that for $n>N$, the last member of (3.8) is greater than the last member of (3.7), so that we have

$$Q_n(x_3) \leqq u_n - u_{n+1} \qquad \text{if } n > N.$$

Hence it follows from the continuity of $Q_n$ in the $L_1[t_n, t_{n+1}]$ topology and the density of $M_c$ in $M$ that there exists an element $x_4 \in M_c$ for which

$$Q_n(x_4) \leqq u_n - u_{n+1} \qquad \text{if } n > N. \tag{3.9}$$

Now if we put

$$x(t) = \lambda x_2(t) + (1 - \lambda)x_4(t), \qquad t \in I,$$

it follows from (3.6) and (3.9) and the continuity of $Q_n(x)$ that there is a value of $\lambda$ on (0, 1) for which (3.4) holds, if $n>N$. Thus for $n>N$, it is possible to choose $x(t)$ on each interval $[t_n, t_{n+1}]$ so as to make (3.4) hold, and these choices can be made independently for each $n>N$. Let such choices be made for each interval $[t_n, t_{n+1}]$, $n>N$, and choose $x$ on the previous intervals in any way which makes $x \in M_c$. Using this choice of $x$, we now define $y$ on $[0, 1)$ by substituting this $x$ in (1.3). It now follows from (3.4) that for $n>N$, (3.5) holds, and from (3.3), (3.5), (1.3) that $y(t_n)$ is independent of $n$ for $n>N$. Setting $y(1)$ equal to this constant value, we have

$$y(t_n) = y(1), \qquad n > N, \tag{3.10}$$

and $y(t)$ is now defined everywhere on $I$.

To show that $y$ is left continuous at $t=1$, we assume $n>N$ and $t_n \leqq t \leqq t_{n+1}$, and write, using (3.10), (1.8), (3.1) and the monotonicity of $x$,

$$\begin{aligned}
|y(t) - y(1)| &= |y(t) - y(t_n)| \\
&\leqq x(t_{n+1}) - x(t_n) + \int_{t_n}^{t_{n+1}} |f[x(s)]|\, ds \\
&\leqq u_{n+1} - u_n + \int_{t_n}^{t_{n+1}} u_{n+1}^{5/3} ds \\
&= [(2n+1)\pi]^{3/4} - (2n\pi)^{3/4} + [2(n+1)\pi]^{5/4}[n^{-1/3} - (n+1)^{-1/3}] \\
&\to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

and it follows that $y \in C$. Moreover, this choice of $y$ corresponds to no solution $x \in C$ of (1.3), since the solution is the chosen $x$ and is unique on each interval $[0, 1-\epsilon]$, and any solution on $[0, 1]$ would have to agree on [0, 1) with this unbounded $x$ that we chose. Thus it is established that if $f$ is given by (1.8), the equation (1.3) does not have a solution for every $y$ in $C$.

4. **Conclusion.** For simplicity, we did not state Theorem 1 in its most general form, and it is easy to see from the proof of the theorem that we can weaken two of the hypotheses a little.

THEOREM 2. *If we weaken conditions* (1.4) *and* (1.5) *of Theorem* 1 *by replacing them by conditions*

$$(1.4') \qquad f(t, u) \operatorname{sgn} u \geqq -A_1 \exp\left(\frac{u^2}{t+\gamma}\right) \qquad \textit{in } R,$$

$$(1.5') \quad f_u(t, u) + 4g_t(t, u) \leqq 2\alpha^2 u^2 + A_2 + \{\max [0, f(t, u) \operatorname{sgn} u]\}^2 \textit{ in } R,$$

*where $\gamma$ is a positive constant, it follows that the conclusion of Theorem* 1 *holds.*

Whether or not Theorem 2 is really more general than Theorem 1 is still an open question, as the author has not yet found any function which satisfies the hypotheses of Theorem 2 but not those of Theorem 1.

In a previous paper, [1], the author raised the question whether

$$y(t) = x(t) + \int_0^t [x(s)]^2 ds$$

has solutions for almost every $y$ in $C$, and he pointed out two other questions which are equivalent to this one. These questions are not answered by this paper, since $f(t, u) \equiv u^2$ does not satisfy condition (1.6). They have, however, recently been answered in the negative in an unpublished paper by D. A. Woodward.

## REFERENCES

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