ISOTOPY IN 3-MANIFOLDS

I. ISOTOPIC DEFORMATIONS OF 2-CELLS AND 3-CELLS¹

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1. Introduction. The author was launched on the investigations of this paper and others to follow by a seminar study of the series of papers by E. E. Moise entitled *Affine Structures in 3-manifolds*. Some of the results are isotopic extensions or versions of corresponding results in that series and the author is indebted to Moise for some of the techniques of proof.

The separable metric space M is an n -manifold (n -manifold with boundary) if every point p of M has a neighborhood homeomorphic to Euclidean n -space, E^n (a neighborhood whose closure is a closed topological n -cell). If M is an n -manifold with boundary, the boundary of M , denoted by $Bd(M)$, is defined to be the set of points of M which do not have a neighborhood in M homeomorphic to E^n . This kind of boundary is to be distinguished from the point set boundary of M in a containing space. The set $M - Bd(M)$ is called the interior of M and is denoted by $Int(M)$. If M is homeomorphic to a locally finite simplicial complex, M is said to be *triangulable*. Moise [5] has shown that every 3-manifold is triangulable and Bing [1] has proved the corresponding result for 3-manifolds with boundary. The same notation will frequently be used for the space and for its triangulation (i.e., the associated complex) wherever no confusion is likely to occur.

Let K and M be triangulated spaces and f a mapping of K into M . Then f is said to be *piecewise linear* if there exist subdivisions K' and M' of K and M , respectively, such that f maps each simplex of K' in a linear (i.e., affine) fashion onto a simplex of M' . If f is in fact a homeomorphism and f' is another such homeomorphism, then a *piecewise linear isotopy* between f and f' is defined to be a mapping $G(x, t)$ of $K \times [0, 1]$ into M such that for all x in K , $G(x, 0) = f(x)$, $G(x, 1) = f'(x)$, and for every t in the closed interval $[0, 1]$, $G(x, t)$ is a piecewise linear homeomorphism. If $K \subset M$ and f is the identity, G is called an isotopic deformation of K . If there is a fixed subdivision K' of K on which $G(x, t)$ is piecewise linear for every t in $[0, 1]$, then

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¹ The first two papers in this series contain the results of the author's thesis written at the University of Wisconsin under the direction of R. H. Bing.

G is said to be *simplicial* (with respect to K'). An isotopy G is *fixed* or is the *identity* on a set S if $G(x, t) = x$ for all points (x, t) in $S \times [0, 1]$. The set S is *invariant* under G if $G(x, t)$ maps S onto itself for all t in $[0, 1]$.

If h is a piecewise linear homeomorphism and F a piecewise linear isotopy defined on the image space of h , then Fh will denote the piecewise linear isotopy $F(h(x), t)$. If h is defined on the image space of F , then hF will denote the piecewise linear isotopy $h(F(x, t))$.

For simplicity and convenience of notation, all cells and simplexes will be considered closed, all complexes locally finite simplicial, and all subdivisions simplicial unless otherwise specified. A set S in a triangulated n -manifold K with boundary is said to be *polyhedral* in K if there is a subdivision K' of K such that S is a union of simplexes of K' .

An n -cell C is said to be *free* in an n -manifold K with boundary which contains it if the intersection $C \cap Bd(K)$ is homeomorphic to an $(n-1)$ -cell. All faces of C in $C \cap Bd(K)$ will be called *free faces* of C . Note that if K is an n -cell and C is free and polyhedral in K , then $Cl[K - C]$ is also an n -cell (for $n \leq 3$ at least).

Let K be an n -manifold with boundary and suppose S is a subset of K . S can be *shelled* from K if it has a cellular decomposition (i.e., a decomposition into n -cells with disjoint interiors) whose cells can be given an ordering $C_1, C_2, C_3, \dots$ (called a *shelling order*) such that C_i is free in $Cl[K - \bigcup_{j < i} C_j]$. If $S = K$, this ordering is called a *shelling* of K .

The concept of shelling is central in the deformation theorems of this paper and is also important in certain characterizations of spaces (see [2], [4], and Theorem 7).

2. The 2-cell. Most of the results of this section are preliminary to those of the next section and will be stated as lemmas.

It has long been known that every triangulated disk can be shelled, e.g. [4], but the more general form of Lemmas 1–3 is required here.

LEMMA 1. *If D is a free 2-cell in a 2-manifold K with boundary and Δ is any cellular decomposition of D , then D can be shelled from K . In particular (setting $D = K$), any cellular decomposition of a disk can be shelled.*

PROOF. Suppose D has m 2-cells under Δ . If $m = 1$, the lemma is trivial, hence the proof will be given for arbitrary m , assuming the lemma for all disks having fewer than m 2-cells.

Since D is free in K , $D \cap Bd(K)$ is an arc ab . Consider the subarcs

which are the intersections of ab with the 2-cells of D . Let the endpoints of these subarcs be ordered from a to b and suppose C is the 2-cell of D containing the first arc aa_1 of ab . If $C \cap Bd(K) = aa_1$, then C is free in K . Otherwise let b_1 be the first endpoint of $C \cap Bd(K)$ between a_1 and b (b_1 may equal b). Then $Bd(C)$ consists of two arcs joining a_1 to b_1 , one of which contains aa_1 , the other lying in $\text{Int}(D)$ except for its endpoints. The latter arc must then cut off a disk D' which is free in both D and K , and has fewer than m 2-cells since it does not contain C . Also $\text{Cl}[D - D']$ is free in $\text{Cl}[K - D']$ and has fewer than m 2-cells since D' is not void. Thus by the induction hypothesis, D' can be shelled from K and then $\text{Cl}[D - D']$ can be shelled from $\text{Cl}[K - D']$ and the lemma is proved.

LEMMA 2. *Every cellular decomposition of a disk has two free 2-cells.*

PROOF. Lemma 2 is an immediate corollary of Lemma 1. Letting $K = D$ in Lemma 1, a free 2-cell C_1 is obtained. Next, replacing the K and D of Lemma 1 by D and $\text{Cl}[D - C_1]$, respectively, a second 2-cell $C_2 \neq C_1$ is obtained which is free in D .

LEMMA 3. *If D and $\text{Cl}[D - K]$ are both disks, where K is a union of 2-cells of D under some cellular decomposition, then K can be shelled from D .*

PROOF. By applying Lemma 2 to the given cellular decomposition of D where $\text{Cl}[D - K]$ is considered as a single 2-cell, a shelling of D can be obtained which leaves $\text{Cl}[D - K]$ till the last.

An open subset of a triangulated space whose closure is polyhedral will be referred to as an *open polyhedron*.

LEMMA 4. *Suppose D is a polyhedral disk in a plane E of E^3 (under a Euclidean, i.e., rectilinear triangulation). Let σ be any 2-simplex of D , and U an open polyhedron containing $(D - \sigma) \cup (Bd(\sigma) \cap \text{Int}(D))$. Then there is a simplicial isotopy G of E^3 onto itself which leaves E invariant, is fixed on $E^3 - U$, and deforms D onto σ .*

PROOF. Let n be the number of 2-simplexes of D . Using induction on n , suppose the lemma is true for all disks with fewer than n 2-simplexes (it is trivially true for $n = 1$). By Lemma 2, there is a free simplex σ' in D which is distinct from σ . Let v_1, v_2 denote the vertices of σ' which are the endpoints of $\sigma' \cap Bd(D)$. The latter set contains either one or two edges of σ' but both cases can be dealt with simultaneously by letting p denote either the third vertex of σ' or the midpoint of v_1v_2 , whichever lies in $Bd(D)$, and denoting the other of the two points by q . Now choose two points v_3, v_4 on pq extended through

p and q , respectively, and two points v_5, v_6 on opposite sides of E , close enough to σ' so that the convex polyhedron P determined by all the v_i lies in U , and $P \cap Bd(D) = \sigma' \cap Bd(D)$. Let $d(x, y)$ be the metric in E^3 and define a piecewise linear isotopy F as follows: F is fixed outside P , $F(p, t)$ is the point of pq a distance $td(p, q)$ from p , and every line joining p to a point x in $Bd(P)$ is mapped linearly onto the line joining $F(p, t)$ to x . Clearly F leaves E invariant, is fixed outside U and deforms D onto $Cl[D - \sigma']$. The induction hypothesis yields a piecewise linear isotopy F' leaving E invariant, fixed outside U and deforming $Cl[D - \sigma']$ onto σ . Then $G = F'F$ is the required isotopy. G is obviously simplicial with respect to some subdivision of E^3 .

LEMMA 5. *Let D and D' be polyhedral disks in the boundary of a tetrahedron T in E^3 such that $D \cap \text{Int}(D')$ is non-null. If U is any open set containing $D \cup D'$, then D' can be deformed onto D by a simplicial isotopy G of E^3 onto itself which leaves $Bd(T)$ invariant and is fixed outside U .*

PROOF. Let A be a triangle in $Bd(T) - D$. One can easily define a piecewise linear homeomorphism f of E^3 onto itself which throws $Bd(T) - A$ into a plane E . Since $D \cap \text{Int}(D')$ is non-null, it may be assumed that D and D' are so triangulated that they have a common simplex σ on which f is linear. Then by Lemma 4, a simplicial isotopy F can be defined which deforms $f(D)$ onto $f(\sigma)$, leaves E invariant, and is fixed on $f(A)$ and outside $f(U)$. Thus $H = f^{-1}Ff$ is a simplicial isotopy which deforms D onto σ , leaves $Bd(T)$ invariant, and is fixed outside U . A similar simplicial isotopy H' can be defined for D' ; hence $G = H^{-1}H'$ satisfies the lemma.

The following theorem is a generalization of Lemma 4 and is proved inductively in a similar manner.

THEOREM 1. *Let D be a polyhedral disk in a triangulated 3-manifold M , σ any 2-simplex of D , and U any open polyhedron in M containing $(D - \sigma) \cup (Bd(\sigma) \cap \text{Int}(D))$. Then D can be deformed onto σ by a simplicial isotopy of M onto itself which is fixed on $M - U$.*

PROOF. Since D is polyhedral in M , it is the sum of, say n 2-simplexes of some subdivision of M . Using induction on n (the case $n = 1$ is trivial), the theorem will be proved for n simplexes, assuming it to be true for all such disks containing fewer than n 2-simplexes.

Let $\sigma' \neq \sigma$ be a free simplex of D . A simplicial isotopy F of D onto $Cl[D - \sigma']$ which is fixed on $M - U$ will be described, which, together with the induction hypothesis, suffices to prove the theorem.

Let p be a vertex of σ' interior to D , if such a vertex exists; other-

wise let p be any point of $Bd(\sigma')$ interior to D . Let $st'(p)$ denote the star of p relative to D (i.e., the union of simplexes of D containing p). There is a vertex v of M such that $st(v) \supset st'(p)$ and S. S. Cairns has shown that every triangulated 3-manifold is a Brouwer manifold; hence there is a homeomorphism h of $st(v)$ into E^3 which is linear over each simplex of $st(v)$ and thus over each simplex of $st'(p)$. Moreover, h can be extended piecewise linearly to an open subset W of M containing $st(v)$ (for example, W could be chosen to be an open 3-cell and h extended to W by means of Lemma 2 of [3]).

If T is a tetrahedron in E^3 containing $h(W)$, then the projection of $Bd(h(st'(p)))$ from $h(p)$ onto $Bd(T)$ is a simple closed polygon J . Since J must bound a polyhedral disk in $Bd(T)$, Lemma 5 can be applied to obtain a piecewise linear homeomorphism g of T onto itself, throwing J onto a polygon in $Bd(T)$ which lies in a plane containing $h(p)$. Now let g' be the piecewise linear homeomorphism of T onto itself which is equal to g on $Bd(T)$ and maps each line joining $h(p)$ to a point q in $Bd(T)$ linearly onto the line joining $h(p)$ to $g(q)$. Then $f = g'h$ throws $st'(p)$ onto a plane polyhedral disk in E^3 in which $f(\sigma')$ is a free simplex. Using the methods of Lemma 4, a simplicial isotopy G can be obtained, which deforms $f(st'(p))$ onto

$$Cl[f(st'(p) - \sigma')]$$

and is fixed outside an open set which lies in $f(U \cap W)$ and contains no point of $f(D - st'(p))$. The required isotopy F is then defined to be the identity on $M - W$ and equal to $f^{-1}Gf$ on W .

If D and M are as in Theorem 1, the conclusion of the theorem implies that there is an open polyhedron V in M whose closure contains D , and a piecewise linear homeomorphism of V into E^3 which throws D into a plane. This, together with Lemma 3, leads to a useful extension of Theorem 1.

COROLLARY 1. *Suppose D and $Cl[D - K]$ are both polyhedral disks in a 3-manifold M , where K is a 2-complex in D . If U is any open polyhedron in M which contains $K - Bd(K \cap Cl[D - K])$, then there is a simplicial isotopy of M onto itself which is fixed on $M - U$ and deforms D onto $Cl[D - K]$. Moreover, if L is a polyhedral 2-manifold in M containing D , the isotopy can be chosen to be invariant on L .*

PROOF. Let D' be a polyhedral disk in L which contains D in its interior. Apply Theorem 1 to obtain an open polyhedron V whose closure contains D' , and a piecewise linear homeomorphism f of V into E^3 , mapping D' into a plane E . Use Lemma 3 to obtain a shelling of $f(K)$ from $f(D)$ and then apply the methods of Lemma 4 to define

a simplicial isotopy G which deforms $f(D)$ onto $f(\text{Cl}[D-K])$, is fixed outside $f(U \cap V)$, and is invariant on E . The desired isotopy is equal to $f^{-1}Gf$ on V and is the identity on $M - V$.

3. The 3-cell. Most of the theorems of this section are three-dimensional analogues of the results of the preceding section.

Examples have been given in the literature of triangulations of 3-cells which cannot be shelled (see [4] and [6]). This means that Theorem 3 is the best possible in the sense that there exist triangulations of 3-cells which must be subdivided before they can be shelled. Cairn's example involves curvilinear simplexes but Frankl's is a Euclidean triangulation. R. H. Bing has also constructed a simple example (unpublished) of a 3-cell in E^3 with a Euclidean triangulation which cannot be shelled. The question remains whether such a triangulation of a tetrahedron exists.

THEOREM 2. *If K is a Euclidean triangulation of a tetrahedron in E^3 , then there is a subdivision K' of K which can be shelled.*

PROOF. The procedure is to cut up the tetrahedron into layers to form a cell complex which can be shelled and then give a consistent triangulation of each cell so that it can be shelled off simplex at a time instead of all at once.

STEP I. Let π be a family of parallel planes such that no two vertices of K lie in the same member of π . The planes of π which contain the vertices of K , together with the planes of π lying, say halfway, between each adjacent pair of the former subdivide K into the desired cell complex.

Let K_i ($i=1, 2, \dots, n$) be the resulting layers ordered from one end of the tetrahedron to the other. Only one K_i will be dealt with at a time; therefore the bounding planes will be denoted by L and L' where L contains a vertex p of K and L' contains none. Since all vertices of K_i lie in L or L' , K_i consists of convex 3-cells which may be of four types. If the 3-cell contains p , it is either

I. A tetrahedron with base in L' ,

II. A hexahedron having four quadrilateral and two triangular faces with a triangular face in L and a quadrilateral face in L' , or

III. A prismoid with triangular bases in L and L' . If it does not contain p , it must be of type III or

IV. A prismoid with quadrilateral bases in L and L' .

STEP II. Now it will be shown that the cell complex K_i can be shelled from $\text{Cl}[K - \cup_{j < i} K_j]$; i.e., that the 3-cells of K_i can be given an ordering $C_1, C_2, \dots, C_n$ such that C_k is free in

$$M_k = \text{Cl} \left[K' - \bigcup_{j < i} K_j - \bigcup_{j < k} C_j \right].$$

Moreover, the ordered described here will be a shelling order for K_i itself. Use will be made of this fact in Corollary 2.

Note first that for $i=1$ or n , K_i needs no further subdivision since it consists solely of tetrahedra which can be shelled in the order given by Lemma 1 for the corresponding bases in $K_i \cap L'$.

For $1 < i < n$, consider the shelling order of the 2-cells of the disk $K_i \cap L$ obtained by Lemma 1. With a slight modification, this will be the desired ordering of the corresponding 3-cells of K_i . The change is required to allow for any type I cells which may be in K_i . If such cells are present, they will be injected into the shelling order as follows: Let C_k be the first type II cell in the ordering given by Lemma 1 and denote by e the edge of C_k in L' which bounds a triangular face. By definition of the ordering, $C_k \cap L$ is free in the disk $\text{Cl}[K_i - \bigcup_{j < k} C_j \cap L]$. It can be shown that C_k is free in M_k unless e separates $\text{Cl}[K_i - \bigcup_{j < k} C_j \cap L']$ into two disks, one of which consists solely of bases of type I cells. The latter disk can be shelled from $\text{Cl}[K_i - \bigcup_{j < k} C_j \cap L']$ by Lemma 1. Hence the corresponding type I cells can all be shelled off, whereupon C_k is free. Applying this argument to each type II cell in order, the desired ordering of the 3-cells of K_i is obtained.

STEP III. The cell complex obtained from K must now be triangulated. The 2-skeleton, K^2 , of this cell complex consists of triangles and convex quadrilaterals. Triangulate K^2 by introducing one diagonal in each quadrilateral. Then the subdivision K' of K is obtained by choosing a point in the interior of each 3-cell not of type I and subdividing it radially (i.e., into simplexes determined by the selected point and the boundary simplexes of the 3-cell).

STEP IV. Now it suffices to prove that for $1 < i < n$, each non-tetrahedral 3-cell C_k of K_i can be shelled simplex at a time from M_k under the triangulation of Step III. Any simplex, σ , of C_k having a face in a free face of C_k can be shelled off first since this face of σ is precisely $\sigma \cap \text{Bd}(M_k)$. Then the remaining simplexes of C_k with faces in $\text{Bd}(M_k)$ can be shelled from $M_k - \sigma$ in the order given by Lemma 3 for $\text{Cl}[\text{Bd}(C_k) - \sigma] \cap \text{Bd}(M_k)$ since the latter complex and $\text{Cl}[\text{Bd}(C_k) - \sigma]$ satisfy the hypotheses for K and D , respectively, in that lemma.

Similarly, since the remaining simplexes of C_k form a cone complex over the disk $C_k \cap \text{Cl}[M_k - C_k]$, they can be shelled off in the order given by Lemma 1 for their faces in $C_k \cap \text{Cl}[M_k - C_k]$. This completes the proof of Theorem 2.

COROLLARY 2. *For any 3-simplex σ of the triangulation K' in Theorem 2, there is a shelling of K' which leaves σ till the last.*

PROOF. Note that the shelling of K' as given in the theorem is symmetric in the sense that the order of the K_i may be reversed. Thus, working from both ends, every layer of K' can be shelled off leaving the one containing σ , say K_i , till the last. Let C_k be the 3-cell of K_i containing σ .

Lemma 2 gives a shelling order for the 2-cells of $K_i \cap L'$ which leaves $C_k \cap L'$ till the last and the corresponding 3-cells of K_i can be shelled from K_i in the same order. Next, any simplex $\sigma' \neq \sigma$ can be shelled from C_k and the rest of C_k can be shelled in the order obtained by Lemma 2 for shelling the remaining 2-cells of $Bd(C_k)$ leaving $\sigma \cap Bd(C_k)$ till the last. The corollary is then proved.

The following theorem will be applied extensively in the deformation theorems to follow.

THEOREM 3. *If S is any 3-cell and K is an arbitrary triangulation of S , then there is a subdivision K' of K which can be shelled.*

PROOF. R. H. Bing proved [1] that any two locally finite simplicial complexes, which are homeomorphic 3-manifolds with boundary, have isomorphic subdivisions (i.e., there is a 1-1 correspondence between simplexes of the subdivisions preserving incidence relations). Hence, since S and a tetrahedron in E^3 are homeomorphic 3-manifolds with boundary, any Euclidean triangulation of such a tetrahedron has a subdivision K_1 which is isomorphic to some subdivision K'' of K . Applying Theorem 2, a subdivision K'_1 of K_1 is obtained which can be shelled. The isomorphism between K_1 and K'' then induces a subdivision K' of K'' which is isomorphic to K'_1 and hence can be shelled. Then K' is the required subdivision of K .

NOTE: Corollary 2 applies to S also. That is, any simplex of K' can be left till the last in the shelling of K' .

Theorem 4 is an extension of the results of Theorems 2 and 3 and is proved by much the same methods.

THEOREM 4. *Let M be a triangulated 3-manifold with boundary and S a free polyhedral 3-cell in M . If K is the triangulation of S induced by a subdivision of M , then there is a subdivision K' of K which can be shelled from M .*

PROOF. As in Theorem 3, there is a subdivision K'' of K and an isomorphism between K'' and a Euclidean triangulation of a tetrahedron in E^3 . This reduces the proof of the theorem to the case where

S is a tetrahedron in E^3 with a Euclidean triangulation K , and M is any 3-manifold with boundary in E^3 such that $D = S \cap \text{Cl}[M - S]$ is a disk. By Lemma 5 there is no loss of generality in assuming that D is a face of S .

The subdivision K' of K will be obtained exactly as in Theorem 2 except the family of planes π will be chosen so that every K_i meets D in a disk. Steps I and III are then essentially unaltered. Also Step IV will need no alteration if Step II can be established. That is to say, if it can be proved that the cell complex formed from K by the planes of π can be shelled from M , then Step IV follows as before. But the only change needed in Step II is to apply Lemma 1 in its more general form.

THEOREM 5. *Let S be a polyhedral 3-cell in a triangulated 3-manifold M , K the triangulation of S induced by a subdivision of M , and σ any 3-simplex of K . Then for any open subset U of M containing S , there is a simplicial isotopy of M onto itself which deforms S onto σ and is fixed on $M - U$.*

PROOF. By Theorem 3, there is a subdivision K' of K which can be shelled and is isomorphic to a Euclidean triangulation of a tetrahedron T in E^3 . This implies the existence of a homeomorphism h of S onto T which is linear over each simplex of the shellable subdivision K' . As in the proof of Theorem 1, h can be extended piecewise linearly to an open subset W of M which contains S .

Let σ' be a 3-simplex of K' contained in σ . Clearly σ' can be deformed onto σ by a simplicial isotopy which is fixed on $M - U$; hence it suffices to exhibit a simplicial isotopy G which deforms S onto σ' and is fixed on $M - U$.

To construct G , first order the 3-simplexes of K' by Corollary 2 so that σ' is the last to be shelled off. Denoting by σ_1 the first simplex to be shelled off, a simplicial isotopy F will be defined which deforms S onto $\text{Cl}[S - \sigma_1]$ and is fixed on $M - U$. Similar isotopies can be defined successively to deform $\text{Cl}[S - \sigma_1]$ onto $\text{Cl}[S - \sigma_1 - \sigma_2]$ etc., and G will be the product of all these isotopies.

The inverse of the homeomorphism h of W into E^3 defines a Euclidean metric $d(x, y)$ on W , and the isotopy F will be defined in terms of this metric. Three cases arise according as the free 3-simplex has one, two, or three free faces. However, these will all be dealt with at once. Let n be the number of free faces of σ_1 . Then there is only one $(n-1)$ -simplex P of σ_1 not lying in $\sigma_1 \cap \text{Bd}(S)$ and a unique $(3-n)$ -simplex Q which is not in $\text{Cl}[S - \sigma_1]$ and is opposite P in the sense that it is the only $(3-n)$ -simplex of σ_1 not intersecting P .

Let p_1 be the centroid of P and q_1 the centroid of Q . Let p_2 be a point on the extension of p_1q_1 through p_1 , close enough to p_1 so that the cone complex K_p over $\sigma_1 \cap Bd(S)$ with p_2 as vertex lies in S . Also let q_2 be a point on the extension of p_1q_1 through q_1 , close enough to q_1 so that the cone complex K_q over $\sigma_1 \cap Bd(S)$ with q_2 as vertex lies in $Cl[U \cap W - S]$. Now define $F(q_1, t)$ to be the point of p_1q_1 a distance $td(p_1, q_1)$ from q_1 , let F be fixed outside $K_p \cup K_q$, and suppose F maps every line joining a point x of $Bd(K_p \cup K_q)$ to q_1 linearly onto the line joining x to $F(q_1, t)$. Then F is the required isotopy and the theorem is proved.

The following two theorems are easy consequences of Theorems 4 and 5.

THEOREM 6. *Let M and S be as in Theorem 4, and suppose M is polyhedral in a triangulated 3-manifold M' with boundary which contains $S - Bd(S \cap Cl[M - S])$ in its interior. Then there is a simplicial isotopy of M' onto itself which is fixed on $Bd(M')$ and deforms M onto $Cl[M - S]$.*

PROOF. Let K be the induced triangulation of S as before. By Theorem 4, there is a subdivision K' of K which can be shelled from M .

The required isotopy is merely the product of a finite sequence of isotopies similar to the F in the proof of Theorem 5, deforming M onto $Cl[M - \sigma_1]$, $Cl[M - \sigma_1]$ onto $Cl[M - \sigma_1 - \sigma_2]$, etc. The only difference is that K_p should be chosen to lie in M rather than in S , and K_q in $Cl[W - S]$ instead of in $Cl[U \cap W - S]$. To obtain the necessary extension of h to an open subset W of M' which contains S , first identify corresponding points of $Bd(M')$ and $Bd(N)$ where N is a homeomorph of M' . In the resulting 3-manifold, h can be extended by Fort's method to an open 3-cell W' containing S and this gives the desired extension of h to an open set $W = W' \cap M'$.

The characterization of the 3-sphere embodied in Theorem 7 is closely allied to Theorems 2 and 3 of [2]. It is stronger in the sense that the decomposition need not satisfy Bing's conditions 3.2 and 3.3 of Theorem 3, and weaker in that a 2-cell is required in place of an arbitrary connected set in condition 3.4. With slight modification of the triangulations of the C_i given below, a triangulation of M could be obtained satisfying the requirements of Bing's Theorem 2 and the rest of the proof would follow from his results. However, it is just as easy to give an independent proof.

THEOREM 7. *A necessary and sufficient condition for a 3-manifold M to be a 3-sphere is that it have a cellular decomposition into a finite*

number of 3-cells C_i ($i=1, 2, \dots, n$) such that for $i=2, 3, \dots, n-1$, C_i is free in $\text{Cl}[M - \bigcup_{j<i} C_j]$.

PROOF. The necessity of the condition is obvious since, if M is a 3-sphere, any 3-simplex of a triangulation of M and the closure of its complement in M constitute such a decomposition.

To establish the sufficiency, simply note that by Theorem 5 of [1], the C_i can be triangulated successively to form a triangulation of M under which each C_i is polyhedral. Then by Theorem 6 of this paper, $\text{Cl}[M - C_1]$ can be deformed isotopically onto $\text{Cl}[M - C_1 - C_2]$, the latter onto $\text{Cl}[M - \bigcup_{j<4} C_j]$, etc. Thus an isotopy of $\text{Cl}[M - C_1]$ onto C_n is obtained, which proves that $\text{Cl}[M - C_1]$ is a 3-cell and hence that M is a 3-sphere.

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