ON THE LAGUERRE METHOD FOR SEPARATING THE ROOTS OF ALGEBRAIC EQUATIONS¹

STEPHEN KULIK

1. Introduction. The celebrated French mathematician E. N. Laguerre gave a proposition [2] which says that any two numbers u and v satisfying the relation

$$(1) \quad (u-x)(v-x)(f'^2 - ff'') + (u+v-2x)ff'' + Nf^2 = 0,$$

where $f \equiv f(x)$, $f(x) = 0$ is an algebraic equation of degree N with real roots and x an arbitrary real number, separate the roots of the equation. This proposition was also discussed by Ch. Hermite [2] and by E. Bodewig [1].

According to the proposition there will be at least one root of the equation between u and v . A generalization of the relation (1), given in the following section gives a method for separating just one root and, in fact, to evaluate it.

2. A method for separating the roots of algebraic equations. Let $f(x) = 0$ be an algebraic equation with real roots a_i , $i = 1, 2, \dots, N$. Consider the expansion of the function $(u-x)(v-x)f'(x)/f(x)$ into the partial fractions

$$(2) \quad \begin{aligned} (u-x)(v-x)f'(x)/f(x) \\ = \sum (u-a_i)(v-a_i)(x-a_i)^{-1} + N(x-u-v) + \sum a_i, \end{aligned}$$

where the sum is taken over all the roots of the equation. After differentiating $n-1$ times both sides of the equation (2) and dividing by $(n-1)!(-1)^{n-1}$, we obtain

$$(3) \quad \begin{aligned} [(u-x)(v-x)D_n + (u+v-2x)fD_{n-1} + f^2D_{n-2}]f^{-n} \\ = \sum (u-a_i)(v-a_i)(x-a_i)^{-n} \quad n \geq 3, \end{aligned}$$

where $f \equiv f(x)$, and

$$(4) \quad D_n = \begin{vmatrix} f' & f & 0 & \dots & 0 \\ f'' & f' & f & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \frac{f^{(n)}}{(n-1)!} & \frac{f^{(n-1)}}{(n-1)!} & \frac{f^{(n-2)}}{(n-2)!} & \dots & f' \end{vmatrix}$$

Presented to the Society, August 22, 1956; received by the editors November 19, 1956 and, in revised form, February 15, 1957.

¹ Sponsored by the Office of Ordnance Research, U. S. Army.

D_n is evaluated recursively, namely

$$(5) \quad D_n = f' D_{n-1} - f'' f D_{n-2}/2! + \cdots + (-f)^{n-2} f^{(n-1)} D_1/(n-1)! \\ + (-f)^{n-1} f^{(n)} D_0/(n-1)!, \quad D_0 = 1,$$

where the second term will be $-f''f$ when $n=2$.

Thus, if we fix $x \neq a_i$, the solutions of the equation

$$(6) \quad (u-x)(v-x)D_n + (u+v-2x)fD_{n-1} + f^2D_{n-2} = 0$$

coincide with the solutions of the equation

$$(7) \quad \sum (u-a_i)(v-a_i)(x-a_i)^{-n} = 0.$$

Now, if n is even and x an arbitrary real number, then u and v which satisfy the equation (7) would be divided by at least one root of the equation $f(x)=0$.

Further, if

$$(8) \quad |x-a_1| < |x-a_i|, \quad i = 2, 3, \dots, N$$

then

$$(9) \quad \lim_{n \rightarrow \infty} u(n) = a,$$

where $u(n) \equiv u$, v is any fixed real number, and $a \equiv a_1$ represents any root of the equation $f(x)=0$.

Solving now equation (6) for u we have the result

$$(10) \quad u = x - f[(v-x)D_{n-1} + fD_{n-2}]/[(v-x)D_n + fD_{n-1}], \\ n = 4, 6, \dots$$

and

$$(11) \quad a = x - \lim_{n \rightarrow \infty} f[(v-x)D_{n-1} + fD_{n-2}]/[(v-x)D_n + fD_{n-1}].$$

Two particular cases are worth noting:

$$(12) \quad u = x - fD_{n-2}/D_{n-1}, \quad \text{when } v = x,$$

and

$$(13) \quad u = x - fD_{n-1}/D_n, \quad \text{when } v = \infty.$$

The Laguerre case can be obtained from (10) by taking $n=2$ and substituting N for D_0 ; however, D_2 and D_1 are still given by (5) with $D_0=1$.

The character of the approximations of u to a with the increase of n can be easily observed in the following two cases:

1. If v and x are not separated by any root of the equation $f(x)=0$, then u and v , and also u and x , will be separated by the root a alone, provided n is sufficiently large. Therefore u will approach a from the side opposite to x with the increase of n . Formula (12) represents a particular case of this rule.

2. If v and x are separated by the root a alone, then u and v will be separated by the root a alone and there will be no root between u and x , provided n is large enough. Therefore u will approach a from the side of x .

A number of similar results could be obtained by the method employed above. We present below one of them without going into the details of its derivation.

$$(14) \quad u = \frac{x - f[(v-x)^2 D_{n-1} + 2(v-x)fD_{n-2} + f^2 D_{n-3}]}{[(v-x)^2 D_n + 2(v-x)fD_{n-1} + f^2 D_{n-2}]} \quad n = 5, 7, \dots$$

$$(15) \quad a = \frac{x - \lim_{n \rightarrow \infty} f[(v-x)^2 D_{n-1} + 2(v-x)fD_{n-2} + f^2 D_{n-3}]}{[(v-x)^2 D_n + 2(v-x)fD_{n-1} + f^2 D_{n-2}]}.$$

In this case u and x will be separated by at least one root of the equation $f(x)=0$ with v an arbitrary real number. Consequently u will approach a from the side opposite to x starting with n sufficiently large.

REFERENCES

1. E. Bodewig, *Sur la méthode de Laguerre pour l'approximation de racines de certaines équations algébriques et sur la critique d'Hermite*, Koninklijke Nederlandsche Akademie Wetenschappen Proceedings, XLIX, No. 8 (1946) p. 911-921.
2. E. N. Laguerre, *Oeuvres de Laguerre* 1, (1898) p. 101 and p. 461.

UNIVERSITY OF SOUTH CAROLINA