A NOTE ON SOME PARTIALLY TRANSITIVE PROJECTIVE PLANES

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1. Introduction. In [3] the author raised the question of the existence of a projective plane of "type $(4, m)$ " with $m=3$, $n \neq 4$. Using results of Baer and Paige [1; 4], we shall show that such a plane does not exist. On the other hand, we will show that a plane of type $(4, m)$ does exist for every $n=p^{2t}$, p a prime.

For background material and definitions the reader is referred to [3] and the references given there; in particular, the notion of a planar ternary ring will be important in what follows. However, we briefly define the class of projective planes of type $(4, m)$. Suppose π is a projective plane of (finite) order n and $\mathcal{G}$ is a collineation group of π , where $\mathcal{G}$ fixes (element-wise) $m \geq 3$ points Q_i , $i=1, 2, \dots, m$, on a line K_0 , a point Q_0 not on K_0 , and the lines K_0 and $K_i=Q_0Q_i$, $i=1, 2, \dots, m$. An ordinary point (line) is a point not on any of the lines K_i (a line not containing any of the points Q_i), $i=0, 1, \dots, m$; suppose furthermore that $\mathcal{G}$ is transitive and regular on the ordinary points and ordinary lines. Then π is of type $(4, m)$. It was shown in [3] that if $m > 3$, then $n=(m-1)^2$; while if $m=3$, $n \neq 4$, then $n=2^t+1$ for some integer t and $\mathcal{G}$ contains a normal subgroup $\mathfrak{R}_0$ which is elementary abelian of exponent two and order 2^t . Furthermore, in the latter case, π is (Q_0, K_0) transitive with group $\mathfrak{R}_0$; this means that $\mathfrak{R}_0$ fixes every point on the line K_0 and every line through the point Q_0 , and that if P_1, P_2 are points of π , neither one on K_0 , neither one equal to Q_0 , such that $Q_0P_1P_2$ is a line, then $P_1x=P_2$ for a unique x in $\mathfrak{R}_0$. Hence (see [3; 5]) any planar ternary ring for π which has $Q_0=(0)$, $K_0=[\infty, (0, 0)]$ will be linear and have its multiplicative loop isomorphic to $\mathfrak{R}_0$. (The point (0) is the intersection of the x -axis with the line at infinity, and $[\infty, (0, 0)]$ is the y -axis.)

2. The nonexistence theorem. We begin this section with a theorem due to Paige [4, Theorem IV.2].

THEOREM 1. *Let (R, F) be a finite linear planar ternary ring with associative and commutative multiplication, and let $e=\Pi x$, for all non-zero x in R . Then $e^2=1$, $xe+x=x+xe=0$ for all x in R , and $xe+(x+y)=y$ for all x, y in R .*

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LEMMA 1. *Let (R, F) be a finite linear planar ternary ring of odd order, with associative and commutative multiplication. If R possesses more than one element of multiplicative order two, then $x + (x + y) = y$ for all x, y in R .*

PROOF. If R possesses more than one element of multiplicative order two, then (see [4], for instance) the element e of Theorem 1 is the identity. Hence the lemma follows from Theorem 1.

Now a duality of a projective plane is a one-to-one mapping of points onto lines, lines onto points, which preserves incidence.

THEOREM 2. *If π is a projective plane coordinatized by a linear planar ternary ring (R, F) with commutative multiplication, in which $x + (x + y) = y$ for all x, y in R , then the mapping T below is a duality of π , of order two:*

$$T: (x, y) \leftrightarrow [x, y]; \quad T: (m) \leftrightarrow [\infty, (m, 0)]; \quad T: (\infty) \leftrightarrow L_\infty.$$

PROOF. If (x, y) is on $[m, k]$, then $mx + y = k$, so $xm + k = mx + k = mx + (mx + y) = y$; i.e., (m, k) is on $[x, y]$. The rest of the proof is trivial.

In [1], Baer has defined an *absolute point* of a duality of order two to be a point which is on its image line, and has shown that if π is a plane of odd order n , with exactly $n+1$ absolute points, then the absolute points form an oval: i.e., a set of $n+1$ points, no three on a line.

THEOREM 3. *If (R, F) is a finite linear planar ternary ring of odd order with associative and commutative multiplication, then R possesses exactly one element of multiplicative order two.*

PROOF. Since the multiplicative group of R has even order, R certainly contains at least one element of multiplicative order two. If there is more than one such element, then by Lemma 1 and Theorem 2, the plane π coordinatized by (R, F) has a duality of order two; it is easy to see that the absolute points of this duality are the $n+1$ points (∞) and $(0, y)$ for all y in R . This set of points is not an oval, so we are done.

COROLLARY. *If π is a projective plane of type $(4, m)$ with $m=3$, then $n=4$.*

PROOF. From the remarks in the introduction, if $n \neq 4$, then π possesses a linear planar ternary ring whose multiplicative loop is an elementary abelian group of exponent two; this is impossible, by Theorem 3.

This completes some of the treatment of type $(4, m)$ in [3], since we now can assert that every plane of type $(4, m)$ has order $n = (m-1)^2$.

3. The existence theorem. Suppose (R, F) is a linear planar ternary ring of order $n = (m-1)^2$, containing a subset R_0 of order $m-1$, and suppose R possesses a group G of mappings satisfying:

- (a) $(x+y)S = xS + yS$, for all x, y in R , all S in G ;
- (b) $(xy)S = (xs)S \cdot yS$, where $sS = 1$, for all x, y in R ;
- (c) $(rs)S = r$, where $sS = 1$, for all r in R_0 ;
- (d) if x, y, u, v in R , x not in R_0y , u not in R_0v , then there is a unique S in G such that $xS = u$, $yS = v$.

Let π be the projective plane coordinatized by (R, F) ; for each S in G , define the mapping S of π as below (where s is the element $sS = 1$):

$$\begin{aligned} S: (x, y) &\rightarrow (xS, yS), & [m, k] &\rightarrow [(ms)S, kS], \\ (m) &\rightarrow ((ms)S), & [\infty, (k, 0)] &\rightarrow [\infty, (kS, 0)], \\ (\infty) &\rightarrow (\infty), & L_\infty &\rightarrow L_\infty \end{aligned}$$

Using (a) and (b), it is easy to see that the set of all such mappings S form a group $\mathfrak{G}$ (isomorphic to G) of collineations of π . From (c) and (d), we deduce that with respect to $\mathfrak{G}$, π is of type $(4, m)$; the fixed points are (∞) , (r) , for all r in R_0 , and $(0, 0)$, while the fixed lines are the lines joining these points.

Now we consider the class of left Veblen-Wedderburn systems (V-W systems, for short) which Hall discovered in [2] (note that Hall defines them as right V-W systems). Let R_0 be a finite field of order $p^t \neq 2$, p a prime, and let $z^2 - az - b$ be irreducible over R_0 . Let R be the set of all elements $\lambda x + y$, for x, y in R_0 , where λ is some indeterminate; define addition and multiplication in R as follows:

- (i) $(\lambda x + y) + (\lambda u + v) = \lambda(x + u) + (y + v)$;
- (ii) $y(\lambda u + v) = \lambda(yu) + yv$;
- (iii) if $x \neq 0$, $(\lambda x + y)(\lambda u + v) = \lambda(au + xv - yu) + yv - x^{-1}u(y^2 - ay - b)$.

As defined above, R is a linear planar ternary ring with the left distributive law and associative addition; R never possesses the right distributive law, and never has associative multiplication except when $p^t = 3$, $a = 0$, $b = -1$ (by the left distributive law, we mean $a(b+c) = ab+ac$). For further discussion of these and similar systems, see [2; 3; 5].

For each choice of d_1, d_2, e_1, e_2 in R_0 , $d_1e_2 - d_2e_1 \neq 0$, define the mapping $S = S(d_1, d_2, e_1, e_2)$ of R by $(\lambda x + y)S = \lambda(d_1x + d_2y) + (e_1x + e_2y)$; let G be the set of all such mappings. Then G is a group, and it is easy to see that G satisfies (a) and (c), where the element s defined by $sS = 1$ is $s = \lambda(-d_2D^{-1}) + d_1D^{-1}$, $D = d_1e_2 - d_2e_1$. The verification of (b) is a matter of straightforward but very tedious computation, which we omit. Finally, let us demonstrate (d).

Let $\lambda x + y$, $\lambda u + v$, $\lambda r + q$, $\lambda w + z$ be elements of R . Then $\lambda x + y$ is in $R_0(\lambda u + v)$ if and only if $xv = yu$. So we assume $xv \neq yu$, and similarly, $rz \neq qw$. Now let $(\lambda x + y)S = \lambda(d_1x + d_2y) + (e_1x + e_2y)$, etc., where S is yet to be determined. If $(\lambda x + y)S = \lambda r + q$, $(\lambda u + v)S = \lambda w + z$, we have:

$$\begin{array}{ll} (1) & \begin{array}{l} d_1x + d_2y = r, \\ d_1u + d_2v = w, \end{array} & (2) & \begin{array}{l} e_1x + e_2y = q. \\ e_1u + e_2v = z. \end{array} \end{array}$$

But (1) and (2) are uniquely solvable for d_1 , d_2 , e_1 , e_2 , if and only if the determinant $xv - yu$ is not zero; this we have by hypothesis. Furthermore, the solutions will satisfy $d_1e_2 - d_2e_1 \neq 0$ if and only if $rz - qw \neq 0$; this we have also assumed.

So for every $n = p^{2^t} \neq 4$, p a prime, there is a plane of type $(4, m)$; in [3] an example with $n = 4$ was also given. It is of interest that the same class of projective planes was used, with a different collineation group, to give an infinite set of examples of planes of type $(5, m)$. The question of the existence of even one projective plane of type $(6, m)$ is still open, however.

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