ON THE SPECTRA OF CERTAIN LAURENT MATRICES¹

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1. Introduction. If G is the set $\{0, \pm 1, \pm 2, \pm 3, \dots\}$, let l_p denote the sequence space $L^p(G)$; the convolution² of two sequences a and b is written $a * b$. When $a = \{a_n\}_{n \in G}$ is in l_1 , then the "Laurent matrix" $(a_{n-\nu})$ represents a transformation $c \rightarrow c * a$ of l_p into itself, which we denote a_{*p} .

Suppose that p is any number such that $p \geq 1$. In this paper we show that, if $a \in l_1$, then the spectrum of a_{*p} is the set $\{A(\theta) : -\pi \leq \theta \leq \pi\}$, where $A(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. I am indebted to Professor R. P. Boas, Jr. for a suggestion which is at the origin of the present note (see §2).

2. Application. Let $\mathfrak{A}$ be the set of all functions analytic on some (variable) open set containing $[-\pi, \pi]$, and denote by ΦA the sequence of Fourier coefficients of A . In the case $p \neq 1$ we have established (see [1]) that, when $a \in \mathfrak{A}$, then $(\Phi A)_{*p}$ is a member $A_{\#p}$ of the set $\mathfrak{E}_p$ of all operators on l_p ; moreover,

(I) *the spectrum of $A_{\#p}$ is the image of $[-\pi, \pi]$ by A .*

It follows directly from the results described in §1 that (I) holds for $p \geq 1$ and when A is in the set $\mathfrak{F}$ of all continuous functions on $[-\pi, \pi]$ having absolutely convergent Fourier series. Thus, if $B(\theta) = [A(\theta)]^{-1}$ defines a function B in $\mathfrak{F}$, then³ the inverse $(A_{\#p})^{-1}$ of $A_{\#p}$ is in $\mathfrak{E}_p$, and $(A_{\#p})^{-1} = B_{\#p}$. This answers affirmatively a conjecture of Professor Boas.

3. Preliminaries. We shall need the following facts. The space l_1 forms a Banach algebra $\mathfrak{L}$ under convolution multiplication and under the norm $\|a\|_1 = \sum_{n=-\infty}^{\infty} |a_n|$. Let δ^n denote the sequence⁴ $\{\delta_\nu^n\}_{\nu \in G}$; the multiplicative unit of $\mathfrak{L}$ is δ^0 . Moreover,

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² We use the notations of [1] and [2, p. 49, 3a].

³ The condition is both necessary and sufficient. Note that B is in $\mathfrak{F}$ if and only if A vanishes nowhere on $[-\pi, \pi]$.

⁴ As usual, $\delta_n^n = 1$ and $\delta_\nu^n = 0$ when $\nu \neq n$.

(α) the set l_0 is dense in $\mathfrak{L}$,

if we agree to denote by l_0 the set of all finite linear combinations of members of $\{\delta^n: n \in G\}$. We shall also consider the space $\mathfrak{C}$ of all functions whose domains include $[-\pi, \pi]$ and which are continuous there; $\mathfrak{C}$ forms a Banach algebra under the multiplication $(A \circ B)(\theta) = A(\theta) \cdot B(\theta)$ and under the norm $\|A\|_\infty = \sup \{|A(\theta)|: -\pi \leq \theta \leq \pi\}$. Suppose that $a \in \mathfrak{L}$; then $a^\wedge$ will denote the function A defined by $A(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, $-\pi \leq \theta \leq \pi$. The mapping $a \rightarrow a^\wedge$ is an isomorphism ψ of $\mathfrak{L}$ onto the subalgebra $l_1^\wedge = \{a^\wedge: a \in l_1\}$ of $\mathfrak{C}$; note that ψ is also a continuous mapping.

3.1. DEFINITION. Suppose that Z is a member of an algebra $\mathfrak{X}$ with unit 1. If X^{-1} denotes the inverse of X when $X^{-1} \in \mathfrak{X}$, then the spectrum $\sigma(Z)$ of Z is the set of all complex λ such that $(\lambda I - Z)^{-1} \notin \mathfrak{X}$ (see [2, p. 64]).

3.2. REMARK. If $T \in \mathfrak{E}_p$, then $\sigma(T)$ is the spectrum of the operator T . If $A \in \mathfrak{C}$, it is easily seen that $\sigma(A)$ is the image $\{A(\theta): -\pi \leq \theta \leq \pi\}$ of $[-\pi, \pi]$ by A . A theorem by Wiener [2, p. 72] states that $\sigma(A) = \sigma(\Phi A)$ when $A \in l_1^\wedge$. In other words, if $a \in l_1$, then $\sigma(a) = \sigma(a^\wedge)$.

3.3. REMARK. The mapping $a \rightarrow a_{*p}$ is a continuous homomorphism of the Banach algebra $\mathfrak{L}$ into the Banach algebra $\mathfrak{E}_p$ (see [2, 32D]).

3.4. LEMMA. If $1 \leq p \leq \infty$ and $a \in l_1$, then $\sigma(a_{*p}) \subset \sigma(a^\wedge)$.

PROOF. It will suffice to show that $\sigma(a_{*p}) \subset \sigma(a)$, since $\sigma(a) = \sigma(a^\wedge)$ by 3.2. To that effect,⁵ let $\lambda \notin \sigma(a)$, whence $(\lambda \cdot \delta^0 - a) * b = \delta^0$; it follows from 3.3 that $(\lambda I - a_{*p})T = I$ for some T in $\mathfrak{E}_p$, whence the conclusion $\lambda \notin \sigma(a_{*p})$.

3.5. LEMMA. If $p = 1$ and $a \in l_1$, then $\sigma(a_{*p}) = \sigma(a^\wedge)$.

PROOF. By 3.4 it will suffice to show that $\sigma(a^\wedge) \subset \sigma(a_{*p})$. To that effect,⁵ let $\lambda \notin \sigma(a_{*p})$, whence $(\lambda I - a_{*p})Tx = x$ for some T in $\mathfrak{E}_1$ and for all x in l_1 ; take $x = \delta^0$ and set $y = Tx \in l_1$, whence $\lambda y - y * a = \delta^0$. But $z \rightarrow z^\wedge$ is a homomorphism, and consequently $(y^\wedge) \circ (\lambda I^0 - a^\wedge) = I^0$, where I^0 is the unit-element of $\mathfrak{C}$. Now $I^0(\theta) = 1$, so that the function $\lambda I^0 - a^\wedge$ must not vanish on $[-\pi, \pi]$; we thus obtain the conclusion $\lambda \notin \sigma(a^\wedge)$ (see 3.2.).

3.6. LEMMA. If $1 \leq p < \infty$ and $a \in l_0$, then $\sigma(a_{*p}) = \sigma(a^\wedge)$.

PROOF. In view of 3.5 and $l_0 \subset l_1$, it will suffice to consider the case $1 < p < \infty$. Note first that the function E^n defined by $E^n(\theta) = e^{in\theta}$ belongs to $\mathfrak{A}$ (see §2; we suppose $n \in G$) and that the sequence $\Phi(E^n)$

⁵ If $\lambda \notin E$ implies $\lambda \notin F$, then $F \subset E$.

of Fourier coefficients of E^n is δ^n (see §3). Now a is a linear combination $\sum \lambda_n \delta^n = \Phi(\sum \lambda_n E^n)$; consequently $a = \Phi A$, where A is some member of $\mathfrak{A}$. From the results of [1] mentioned in §2 we obtain $\sigma(A_{\#p}) = \sigma(A)$; but $A_{\#p} = (\Phi A)_{*p} = a_{*p}$ whence $\sigma(A) = \sigma(a_{*p})$. The proof is concluded by noting that $A = a^\wedge$.

3.7. THEOREM. *If $a \in l_1$, then $\sigma(a_{*p}) = \sigma(a^\wedge)$ for any p such that $1 \leq p < \infty$.*

PROOF. Let X_n and Z be members of a commutative Banach algebra $\mathfrak{X}$; if $\|X\|$ is the norm of X in $\mathfrak{X}$, we write $Z = \mathfrak{X} \text{lm} X_n$ whenever $\lim \|Z - X_n\| = 0$. We shall need the following result:

(β) *if $Z = \mathfrak{X} \text{lm} X_n$, then $\sigma(Z) = \text{Lim } \sigma(X_n)$.*

In this statement (proved in [3, p. 169]), Lim is the Hausdorff limit.

If $a \in l_1 = \mathfrak{L}$, then (by (α)) there exist members $a(n)$ of l_0 such that

(1) $a = \mathfrak{L} \text{lm} a(n) \quad \text{and} \quad a(n) \in \mathfrak{L}$

(we suppose $n \rightarrow \infty$ throughout); by combining (1) with 3.3 we obtain $a_{*p} = \mathfrak{L}_p \text{lm}(a(n))_{*p}$. In view of (β), this last result yields $\sigma(a_{*p}) = \text{Lim } \sigma(a(n)_{*p})$; but $a(n) \in l_0$, and by 3.6 we can write the last equality in the form

(2) $\sigma(a_{*p}) = \text{Lim } \sigma(a(n)^\wedge).$

On the other hand, since $a \rightarrow a^\wedge$ is a continuous mapping of $\mathfrak{L}$ into $\mathfrak{L}$, we can infer from (1) that $a^\wedge = \mathfrak{L} \text{lm}(a(n))^\wedge$, together with $a(n) \in \mathfrak{L}$. Combining again with (β), we obtain $\sigma(a^\wedge) = \text{Lim } \sigma((a(n))^\wedge)$ and a comparison with (2) gives the conclusion.

4. Concluding remarks. From 3.2 and 3.7 we see that $\sigma(a_{*p}) = \sigma(a^\wedge) = \sigma(a)$, whence the assertions of §1. On the other hand,

(II) *if $A \in l_1^\wedge$, then $\sigma(A_{\#p}) = \sigma(A)$ whenever $1 \leq p < \infty$.*

This can be seen as follows: the sequence $a = \Phi A$ of Fourier coefficients of A is in l_1 , and $A_{\#p} = a_{*p}$ (see §2); but $a^\wedge = A$, so that (II) follows from 3.7. Keeping 3.2 in mind, it is clear that (II) covers the assertions of §2 (it is easily verified that $l_1^\wedge = \mathfrak{F}$). From (II) it can readily be inferred that

$$\|A\|_\infty = \lim_{n \rightarrow \infty} \|(A_{\#p})^n\|^{1/n} \quad (1 \leq p < \infty).$$

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CONTINUITY PROPERTIES OF DERIVATIVES OF SEQUENCES OF FUNCTIONS¹

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In a recent paper Dvoretzky [1] discusses an interesting generalization of a theorem of Walsh [2]. A striking supplement to Dvoretzky's theorem is the following one.

THEOREM. *There exists a sequence of functions*

$$\{f_n\}_1^\infty, f_n \in C^1(-\infty, \infty),$$

with $\lim_{n \rightarrow \infty} f_n(x) \equiv 0$, such that: if N_1 is any subsequence of the natural numbers with the property that there exists a sequence x_{n_1} , $n_1 \in N_1$, satisfying

$$(1) \quad f'_{n_1}(x_{n_1}) = 0, \quad \text{and} \quad \lim_{n_1 \rightarrow \infty} x_{n_1} = 0,$$

then the sequence N_2 complementary to N_1 (i.e., N_2 contains exactly those natural numbers omitted by N_1) is infinite and

$$(2) \quad \limsup_{n_2 \rightarrow \infty} \int_0^h |f'_{n_2}(x)| dx = \infty$$

for every $h > 0$.

PROOF. Let $\epsilon_n \downarrow 0$ and let $\{\lambda_n\}$ be the sequence 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, $\dots$. Let $\{\mu_n\}$ be a sequence of positive numbers such that

$$(3) \quad \epsilon_n \mu_n / \lambda_n \rightarrow \infty, \quad n \rightarrow \infty.$$

The functions $f_n(x)$ shall be odd and

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