NONCOUNTABLE NORMALLY LOCALLY FINITE DIVISION ALGEBRAS

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A (commutative) field F is *regular* (see [1] of the bibliography) if it is not finite, and if in addition it is true that the direct (=Kronecker =tensor) product of two normal (=central) division algebras, of finite orders, over F is not a division algebra unless their orders are relatively prime; algebraic number fields and p-adic fields are examples of regular fields. A division algebra $\mathfrak{A}$ over a field F is *normally locally finite* if any finite subset of $\mathfrak{A}$ is contained in a normal (over F) division sub-algebra of $\mathfrak{A}$ of finite order; in [1], such algebras were called "of type 1." A *subisomorphism* of an algebra $\mathfrak{A}$ over F is an algebra-isomorphism of $\mathfrak{A}$ into $\mathfrak{A}$, and it is *proper* if it is not onto. If $\mathfrak{A}$ is a normally locally finite division algebra over the regular field F, without a finite basis over F, a *characteristic sub-algebra* of $\mathfrak{A}$ is any normally locally finite division sub-algebra $\mathfrak{D}$ of $\mathfrak{A}$, with countably infinite basis over F, having the property that any normally locally finite division sub-algebra of $\mathfrak{A}$, with finite or countable basis over F, is isomorphic to a sub-algebra of $\mathfrak{D}$. It was proved in [1] that any $\mathfrak{A}$ of the previous type has a characteristic sub-algebra, unique but for isomorphisms; it was also proved that there exists a normally locally finite division algebra over the regular field F, with infinite noncountable basis, and with a given characteristic sub-algebra $\mathfrak{D}$, if and only if $\mathfrak{D}$ admits proper subisomorphisms; [1] contains a rather involved proof of the fact that any $\mathfrak{D}$ admits proper subisomorphisms if F is not countable, and thus establishes the existence of normally locally finite division algebras, with infinite noncountable basis, over any noncountable regular field; this seems to be the only known example of such algebras. We shall present here a very simple proof of the same result, and will, at the same time, dispense with the condition of noncountability of F.

(1). LEMMA. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be normal division algebras, of finite orders >1, over the (certainly infinite) field F, and suppose $\mathfrak{A}\times\mathfrak{B}\times\mathfrak{C}$ also to be a division algebra; let m be an element of $\mathfrak{A}\times\mathfrak{B}$ but not of $\mathfrak{A}$. Then there exists a $d\in\mathfrak{B}\times\mathfrak{C}$, not zero, such that $d^{-1}md\notin\mathfrak{A}\times\mathfrak{B}$.*

In the previous statement, as in the rest of this paper, the identity

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elements of the direct factors of a direct product of algebras are assumed to be coincident.

PROOF (being a modification, due to D. Zelinsky, of a proof by the author). Let c be an element of $\mathfrak{C}$, but not of F, and let b be an element of $\mathfrak{B}$ such that $mb \neq bm$; such b exists because the commutator (=centralizer) of $\mathfrak{B}$ in $\mathfrak{A}\times\mathfrak{B}$ is $\mathfrak{A}$, and $m \notin \mathfrak{A}$. Set $x = bc \neq 0$, so that also $1+x \neq 0$; if the lemma is false, we have $xy = mx$, $(1+x)z = m(1+x)$ for suitable elements y, z of $\mathfrak{A}\times\mathfrak{B}$; subtracting, we obtain $m = z + x(z-y)$. If $y = z$, then $m = z = y$, $xm = mx$, $bm = mb$, a contradiction; if $y \neq z$, then $bc = x = (m-z)(z-y)^{-1} \in \mathfrak{A}\times\mathfrak{B}$, also a contradiction, since $c \notin F$, Q.E.D.

For the convenience of the reader, we repeat here a portion of the statement of (6) of [1]:

(2). LEMMA. *Let $\mathfrak{A}$ be a normally locally finite division algebra, with countably infinite basis, over the field F; a necessary and sufficient condition in order that $\mathfrak{A}$ admit a proper subisomorphism is that there exist a factorization*

$$\mathfrak{A} = \mathfrak{B}_0 \times \mathfrak{B}_1 \times \cdots$$

of $\mathfrak{A}$ as a direct product of normal division algebras of finite orders >1 over F, an $m \in \mathfrak{B}_0$, and a sequence $h_1, h_2, \cdots$ of elements of $\mathfrak{A}$, such that, after setting $\mathfrak{A}_i = \mathfrak{B}_0 \times \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_i$, the following conditions be satisfied:

(a) $h_i \in \mathfrak{A}_i$;

(b) $h_{i+1} = h_i c_i$ *for a* $c_i \in \mathfrak{B}_i \times \mathfrak{B}_{i+1}$;

(c) *there exists no* $z_{i-1} \in \mathfrak{A}_{i-1}$ *such that* $h_i z_{i-1} = m h_i$.

We can now prove:

(3). THEOREM. *Let $\mathfrak{A}$ be as in* (2); *then $\mathfrak{A}$ admits a proper subisomorphism.*

PROOF. By (29) of [1],[1] $\mathfrak{A}_0$ cannot be transformed into itself by every inner automorphism of $\mathfrak{A}_1$; hence there exist an $m \in \mathfrak{A}_0$, and an $h_1 \in \mathfrak{A}_1$, with $h_1 \neq 0$, such that $h_1^{-1} m h_1 \notin \mathfrak{A}_0$. We shall now proceed to build the sequence $\{h_i\}$ of (2) by induction: assume the $h_1, \cdots, h_i$ to have been found; by (1) (after replacing $\mathfrak{A}$ by $\mathfrak{A}_{i-1}$, $\mathfrak{B}$ by $\mathfrak{B}_i$, $\mathfrak{C}$ by $\mathfrak{B}_{i+1}$, m by $h_i^{-1} m h_i$), there exists a $c_i \in \mathfrak{B}_i \times \mathfrak{B}_{i+1}$, not zero, such that $c_i^{-1}(h_i^{-1} m h_i) c_i \notin \mathfrak{A}_i$. Then $h_{i+1} = h_i c_i$ satisfies the conditions of (2), Q.E.D.

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[1] This is the little theorem with a distinguished career, first proved in [2], which later came to be known as the Cartan-Brauer-Hua theorem (see for instance [**3**, Chapter VII, §13]); the proof given in [1] is the first elementary proof for the finite case.

(4). COROLLARY. *Let* $\mathfrak{A}$ *be as in* (2), *and assume* F *to be regular; then there exists a normally locally finite division algebra over* F, *with infinite noncountable basis, having* $\mathfrak{A}$ *as characteristic sub-algebra.*

On the other hand, if F is not regular, the concept of characteristic sub-algebra loses meaning; however, from (3), and from a slight modification of the construction used to prove the sufficiency of (3) of [1], we still obtain:

(5). COROLLARY. *Let* F *be a field such that there exists a normally locally finite division algebra* $\mathfrak{A}$ *with countably infinite basis over* F; *then there exists a normally locally finite division algebra over* F, *with infinite noncountable basis, having* $\mathfrak{A}$ *as a sub-algebra.*

REMARK. An examination of the proof of (3) of [1] discloses that all the normally locally finite division algebras, with infinite noncountable basis over F, whose existence has been established in this note, have a basis of cardinality $\aleph_1$; the existence of normally locally finite division algebras over F, with a basis of cardinality $>\aleph_1$, is still an open problem, at least when F is regular.

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