NOTE ON A PAPER OF DIEUDONNÉ

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In a recent paper [1], Dieudonné has proved the following theorem.

Let $a_0, a_1, a_2, \dots$ be an infinite sequence of rational numbers and consider the (formal) power series

$$\exp\left(\sum_{m=0}^{\infty} a_m x^{p^m}\right) = \sum_{n=0}^{\infty} c_n x^n,$$

where p is a fixed prime. Then a necessary and sufficient condition that all the coefficients c_n be p -adic integers is that for each $i \geq 0$ we have

$$a_i = \frac{a_{i-1}}{p} + b_i \quad (a_{-1} = 0),$$

where each b_i is a p -adic integer.

In this note we consider the following problem. Let $a_1, a_2, a_3, \dots$ be a sequence of rational numbers and define the (formal) power series

$$(1) \quad \exp\left(\sum_{m=1}^{\infty} a_m x^m\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Then we prove the following

THEOREM 1. *A necessary and sufficient condition that all the coefficients c_n be rational integers is that for all $k \geq 1$ we have*

$$(2) \quad \sum_{rs=k} r a_r \mu(s) \equiv 0 \pmod{k},$$

where $\mu(s)$ is the Möbius function.

To prove the theorem we employ a device used by Schur [2] and credited to Jänichen. Let

$$g(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Then we can recursively determine $b_1, b_2, \dots$ so that

$$(3) \quad g(x) = \prod_{m=1}^{\infty} (1 - x^m)^{b_m}.$$

Moreover the b_m are all integral if and only if the c_n are all integral. Now from (1) and (3) it follows that

$$\begin{aligned}\sum_{k=1}^{\infty} a_k x^k &= \log g(x) = - \sum_{m=1}^{\infty} b_m \sum_{r=1}^{\infty} \frac{x^{mr}}{r} \\ &= - \sum_{k=1}^{\infty} \frac{x^k}{k} \sum_{mr=k} m b_m;\end{aligned}$$

consequently

$$(4) \quad k a_k = - \sum_{m|k} m b_m.$$

But (4) is equivalent to

$$(5) \quad -b_k = \frac{1}{k} \sum_{rs=k} r a_r \mu(s).$$

As already noted, the c_n are all integral if and only if the b_k are all integral, that is, if and only if the right member of (5) is integral for all k . But this is the same as the condition (2). This evidently proves the theorem.

The theorem can be stated in a slightly more general way. Let P denote an arbitrary set of rational primes and let D denote the set of rational numbers whose denominators contain only the primes of P . Then we may state

THEOREM 2. *A necessary and sufficient condition that all the coefficients $c_n \in D$ is that for all $k \geq 1$*

$$\left\{ \frac{1}{k} \sum_{rs=k} r a_r \mu(s) \right\} \in D.$$

Dieudonné's theorem is obtained when P consists of all the primes except p and $a_m = 0$ except for $m = p^r$.

REFERENCES

1. J. Dieudonné, *On the Artin-Hasse exponential series*, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 210-214.
2. I. Schur, *Arithmetische Eigenschaften der Potenzsummen einer algebraischen Gleichung*, Compositio Math. vol. 4 (1937) pp. 432-444.

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