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ON A PROBLEM OF D. R. HUGHES

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D. R. Hughes (Bull. Amer. Math. Soc. vol. 63 (1957) p. 209) has proposed the following problem:

Let G be a group and p a prime. Define $H_p(G)$ to be the [normal] subgroup of G generated by all the elements of G which do not have order p . Is the following conjecture true: either $H_p(G) = \{1\}$ or $H_p(G) = G$ or $[G: H_p(G)] = p$?

He remarks that the conjecture is true for $p=2$.

In this note we prove Hughes' conjecture for $p=3$.

We shall use the following notation: if $h, g_1, \dots, g_n \in G$ and $a_1, \dots, a_n$ are integers then

$$h^{a_1 g_1 + \dots + a_n g_n} = g_1^{-1} h^{a_1} g_1 g_2^{-1} h^{a_2} g_2 \dots g_n^{-1} h^{a_n} g_n.$$

LEMMA 1. If $h \in H_p$, $x \notin H_p$ then

$$h^{1+x+x^2+\dots+x^{p-1}} = 1.$$

PROOF. Since $x^{p-1} \notin H_p$, all elements of $H_p x^{p-1}$ have order p . In particular

$$\begin{aligned} 1 &= (hx^{p-1})^p = hx^{p-1} \cdot hx^{p-1} \dots hx^{p-1} \\ &= h \cdot x^{-1} h x \cdot x^{-2} h x^2 \dots x^{-(p-1)} h x^{p-1} = h^{1+x+\dots+x^{p-1}}. \end{aligned}$$

LEMMA 2. If $h \in H_3$ and $xH_3 \neq yH_3$ then $h^{x+y} = h^{y+x}$.

PROOF. By hypothesis $z = x^{-1}y \notin H_3$. Hence by Lemma 1

$$1 = h_1^{1+z+z^2} = h_1^{1+z^2+z} \quad h_1 \in H_3$$

or $h_1^{z+z^2} = h_1^{z^2+z}$. Setting $h_1 = h^{xz^2}$ proves the lemma.

THEOREM. *If $[G: H_3] > 3$ then $H_3 = \{1\}$.*

PROOF. Since all elements of G/H_3 are of order 3, every finitely generated subgroup of G/H_3 is finite by Burnside's theorem. In particular, therefore, G/H_3 has an Abelian subgroup of order 9. Let such a subgroup be generated by xH_3, yH_3 . We have

$$\begin{aligned} 3 = & -(1+x+x^2)y - y^2(1+x+x^2) + (1+y+y^2) \\ & + (1+xy+y^2x^2) + (1+x^2y+y^2x) = f(x, y). \end{aligned}$$

But according to Lemma 2 we have therefore $h^3 = h^{f(x,y)}$ for every $h \in H_3$, and by Lemma 1 this implies $h^3 = 1$. In other words H_3 contains only the identity.

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