ON THE PRODUCT OF LEGENDRE AND BESSEL FUNCTIONS

F. M. RAGAB

1. Introduction. The following formulae will be established: If $|\arg z| < 3\pi/2$, $R(1-m) > 0$,

$$(1) \quad \int_0^\infty \lambda^{-m/2} (1+\lambda)^{(m-\nu)/2} K_\nu(z(1+\lambda)^{1/2}) P_n^m(1+2\lambda) d\lambda \\ = \frac{z^{\nu-2m-2}}{2^{\nu-2m} \pi} \sum_{i, -i} \frac{1}{i} E(1+n+m, m-n, m-\nu+1; : e^{i\pi} \cdot z^2/4)$$

where $\sum_{i, -i}$ denotes that to the expression following it a similar expression with i_1-i interchanged is to be added. If $R(1/4+\nu/2-n) > 0$, $R(5/4+\nu/2+n) > 0$, $R(1-m) > 0$ and z is real and positive.

$$(2) \quad \int_0^\infty \lambda^{-m/2} (1+\lambda)^{m/2-\nu/2} J_\nu(z(1+\lambda)^{1/2}) P_n^m(1+2\lambda) d\lambda \\ = \frac{\Gamma(2m+1)}{\Gamma(n-m+1)\Gamma(\nu-n)} \left(\frac{1}{2} z\right)^{\nu-2n-2} {}_1F_2\left(\begin{matrix} m-n; -z^2/4 \\ \nu-n, -2n \end{matrix}\right) \\ + \frac{\Gamma(-2n-1)}{\Gamma(-m-n)\Gamma(1+n+\nu)} \left(\frac{1}{2} z\right)^{2n+\nu} {}_1F_2\left(\begin{matrix} 1+m+n; -z^2/4 \\ 1+n+\nu, 2+2n \end{matrix}\right).$$

These two formulae will be deduced from more general formulae [(9) and (14) below] by means of the formula

$$(3) \quad P_n^m(1+2\lambda) \\ = \frac{1}{\Gamma(1-m)} \{\lambda(1+\lambda)\}^{-m/2} F\left(\begin{matrix} -m-n, n-m+1; -\lambda \\ 1-m \end{matrix}\right).$$

For the definition of the E -functions, the reader is referred to (T. M. MacRobert, *Functions of a complex variable*, 4th ed., Glasgow, 1954, p. 352). Formula (1) will be proved in §3 by means of the following subsidiary formula

$$(4) \quad \frac{2\pi z^{\alpha+\beta-1} e^{-z}}{\Gamma(1-\alpha)\Gamma(1-\beta)} E(1-\alpha, 1-\beta; : z) = \sum_{i, -i} \frac{1}{i} E(\alpha, \beta; : e^{i\pi} z),$$

which will be proved in §2. (2) will be proved in §4. The following formulae will be required in the proofs:

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$$(5) \quad E(p; \alpha_r; q; \rho_s; z) = \prod_{s=1}^p \Gamma(\alpha_s - \alpha_r) \left\{ \prod_{t=1}^q \Gamma(\rho_t - \alpha_r) \right\}^{-1} \Gamma(\alpha_r) \\ \times z^{\alpha_r F} \left\{ \alpha_r, \alpha_r - \rho_1 + 1, \dots, \alpha_r - \rho_q + 1; \quad (-1)^{p-q} z \right\} \\ \left\{ \alpha_r - \alpha_1 + 1, \dots, \alpha_r - \alpha_p + 1 \right\};$$

$$(6) \quad \int_0^\infty e^{-(t+z^2/t)/2} t^{-\nu-1} dt = 2z^{-\nu} K_\nu(z),$$

where $R(z) > 0$, $R(z^2) > 0$;

$$(7) \quad \int_0^\infty e^{-\lambda z} \lambda^{\gamma-1} F\left(\begin{matrix} \alpha, \beta; -\lambda \\ \gamma \end{matrix}\right) d\lambda = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} z^{-\gamma} E(\alpha, \beta; :z);$$

where $R(\gamma) > 0$, $R(z) > 0$; and

$$(8) \quad \int_0^\infty e^{-\lambda} \lambda^{\alpha_p+1-1} E(p; \alpha_r; q; \rho_s; z/\lambda) d\lambda = E(p+1; \alpha_r; q; \rho_s; z),$$

where $R(\alpha_{p+1}) > 0$.

These formulae are to be found in the book referred to on pp. 353, 389, 393 and 394 respectively.

2. Proof of the subsidiary formula. To prove (4) expand each E -function on the right hand side of (4) by means of (5), add similar terms, so getting

$$E(\alpha, \beta; :e^{i\pi} z) - E(\alpha, \beta; :e^{-i\pi} z) = 2\pi i \\ \times \left[\frac{\Gamma(\beta - \alpha)}{\Gamma(1 - \alpha)} z^\alpha F\left(\begin{matrix} \alpha; -z \\ 1 + \alpha - \beta \end{matrix}\right) + \frac{\Gamma(\alpha - \beta)}{\Gamma(1 - \beta)} z^\beta F\left(\begin{matrix} \beta; -z \\ 1 + \beta - \alpha \end{matrix}\right) \right] \\ = \frac{(2\pi i) z^{\alpha+\beta-1}}{\Gamma(1 - \alpha)\Gamma(1 - \beta)} e^{-z} \cdot \sum_{\alpha, \beta} \Gamma(\alpha - \beta) \Gamma(1 - \alpha) z^{1-\alpha} F\left(\begin{matrix} 1 - \alpha; z \\ 1 + \beta - \alpha \end{matrix}\right),$$

where we have used the well known formula $F(\alpha; \rho; z) = e^z F(\rho - \alpha; \rho; -z)$. If we apply (5) again, formula (4) is obtained.

3. A generalization of formula (1). The formula to be proved is

$$(9) \quad \int_0^\infty \lambda^{\gamma-1} (1 + \lambda)^{-\nu/2} K_\nu(z(1 + \lambda)^{1/2}) F(\alpha, \beta; \gamma; -\lambda) d\lambda = \Gamma(\gamma) \\ \times (z/2)^{2\alpha+2\beta+\nu-2\gamma-2} \\ \cdot \frac{1}{4\pi} \sum_{i, -i} \frac{1}{i} E(1 - \alpha, 1 - \beta, 1 + \gamma - \nu - \alpha - \beta; :e^{i\pi} \cdot z^2/4)$$

where $R(\gamma) > 0$, $|\text{amp } z| < \pi$.

On substituting from (6) on the left hand side of (9) and changing the order of integration it becomes

$$\frac{1}{2} z^\nu \int_0^\infty e^{-t/2 - z^2/2t} \cdot t^{-\nu-1} dt \int_0^\infty e^{-\lambda z^2/2t} \cdot \lambda^{\gamma-1} F\left(\begin{matrix} \alpha, \beta; -\lambda \\ \gamma \end{matrix}\right) d\lambda.$$

From (7), it follows that the inmost integral is equal to

$$\Gamma(\gamma) \{ \Gamma(\alpha) \Gamma(\beta) \}^{-1} (z^2/4t)^{-\gamma} E(\alpha, \beta; : z^2/2t).$$

Thus on replacing t by $2t$, the expression reduces to

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} 2^{2\gamma-\nu-1} z^{\nu-2\gamma} \cdot \int_0^\infty e^{-t - z^2/4t} t^{\gamma-\nu-1} E\left(\alpha, \beta; : \frac{z^2}{4t}\right) dt.$$

Here substitute for the last E -function from (4); and the expression becomes

$$\Gamma(\gamma) (z/2)^{2\alpha+2\beta+\nu-2\gamma-2} \times \sum_{i,-i} \frac{1}{4\pi i} \int_0^\infty e^{-t} t^{\gamma-\nu-\alpha-\beta} E(1-\alpha, 1-\beta; : e^{i\pi} z^2/4t) dt,$$

and from this by applying (8), (9) follows.

SPECIAL CASES. In (9), take $\alpha = -m-n$, $\beta = -m+n+1$, $\gamma = 1-m$, apply (3) and so obtain (1).

Next, by expanding each E -function on the right hand side of (9) by means of (5); it can be written in the form

$$(10) \quad \int_0^\infty \lambda^{\gamma-1} (1+\lambda)^{-\nu/2} K_\nu(z(1+\lambda)^{1/2}) F(\alpha, \beta; \gamma; -\lambda) d\lambda = \frac{\Gamma(\gamma)}{4\pi} \times \sum_{i,-i} \frac{1}{i} E \left[\begin{matrix} -\frac{1}{2} \nu, \alpha - \gamma + \frac{1}{2} \nu, \beta - \gamma + \frac{1}{2} \nu, 1 : e^{i\pi} \cdot z^2/4 \\ \alpha + \beta - \gamma + \frac{1}{2} \nu \end{matrix} \right]$$

where $R(\gamma) > 0$, $|\text{amp } z| < \pi$.

In (10) take $\beta = 0$, $\gamma = \alpha + \nu/2 - 1$; then if $R(\alpha + \nu/2 - 1) > 0$, $R(z) > 0$

$$(11) \quad \int_0^\infty \lambda^{\alpha+\nu/2-2} (1+\lambda)^{-\nu/2} K_\nu(z(1+\lambda)^{1/2}) d\lambda = \Gamma\left(\alpha + \frac{1}{2} \nu - 1\right) \times \sum_{i,-i} \frac{1}{4\pi i} E\left(1-\alpha, 1, -\frac{1}{2} \nu; : e^{i\pi} \cdot z^2/4\right).$$

But, by expanding the following E -functions by means of (5), it can be seen that

$$(12) \quad \sum_{i,-i} \frac{1}{4\pi i} E\left(1 - \alpha, -\frac{1}{2} \nu, 1 :: e^{i\pi z^2/4}\right) = (z/2)^{-\alpha-\nu/2+1} \\ \times \sum_{i,-i} \frac{1}{4\pi i} E\left(\frac{1}{2} - \frac{1}{2} \alpha + \frac{1}{4} \nu, -\frac{1}{4} \nu + \frac{1}{2} \alpha - \frac{1}{2} :: \frac{e^{i\pi z^2}}{4}\right) \\ = \left(\frac{z}{2}\right)^{1-\alpha-\nu/2} K_{1-\alpha+\nu/2}(z).$$

In (11) take $\alpha = 2 + \mu - \nu/2$ and apply (12), so getting

$$(13) \quad \int_0^\infty \frac{K_\nu(z(1+\lambda)^{1/2})}{(1+\lambda)^{\nu/2}} \lambda^\mu d\lambda = \frac{2^{\mu+1}}{z^{\mu+1}} \Gamma(\mu+1) K_{\nu-\mu+1}(z),$$

where $R(\mu) > -I$, $R(z) > 0$.

This formula was given by G. N. Watson (*Theory of Bessel Functions*, Cambridge, 1944, p. 417, formula (6)).

4. A generalization of (2). The formula to be proved is

$$(14) \quad \int_0^\infty \lambda^{\gamma-1} (1+\lambda)^{-\nu/2} J_\nu(z(1+\lambda)^{1/2}) F(\alpha, \beta; \gamma; -\lambda) d\lambda = \Gamma(\gamma) \\ \times \sum_{\alpha, \beta} \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta) \Gamma(1 + \alpha - \gamma + \nu)} \left(\frac{z}{2}\right)^{2\alpha-2\gamma+\nu} \\ {}_1F_2\left(\begin{matrix} 1 - \beta; -z^2/4 \\ 1 + \alpha - \gamma + \nu, 1 + \alpha - \beta \end{matrix}\right),$$

where $R(\gamma) > 0$, $R(5/4 - \gamma + \nu/2 + \alpha) > 0$, $R(5/4 - \gamma + \nu/2 + \beta) > 0$ and z is real and positive.

PROOF OF (14). In (10), let it be assumed that $R(\gamma) > 0$, $R(5/4 - \gamma + \nu/2 + \alpha) > 0$, $R(5/4 - \gamma + \nu/2 + \beta) > 0$ and let $\arg z$ decrease by $\pi/2$, so that z/i is written for z , then, since $K_\nu(t) = i^\nu G_\nu(it)$, we have

$$\int_0^\infty \lambda^{\gamma-1} (1+\lambda)^{-\nu/2} G_\nu(z(1+\lambda)^{1/2}) F(\alpha, \beta; \gamma; -\lambda) d\lambda = \frac{\Gamma(\gamma)}{4\pi i} i^{-\nu} \\ \times \left\{ E\left(-\frac{1}{2} \nu, \alpha - \gamma + \frac{1}{2} \nu, \beta - \gamma + \frac{1}{2} \nu, 1 :: \alpha + \beta - \gamma + \frac{1}{2} \nu : z^2/4\right) \right. \\ \left. - E\left(-\frac{1}{2} \nu, \alpha - \gamma + \frac{1}{2} \nu, \beta - \gamma + \frac{1}{2} \nu, 1 :: \alpha + \beta - \gamma + \frac{1}{2} \nu : e^{-2i\pi} \cdot z^2/4\right) \right\}.$$

Now expand each E -function on the right hand side of the last expres-

sion by means of (5), add applying the formula

$$(15) \quad \Gamma(z)\Gamma(1-z) = \pi/\sin \pi z;$$

so getting if $R(\gamma) > 0$

$$(16) \quad \int_0^\infty \lambda^{\gamma-1}(1+\lambda)^{-\nu/2} G_\nu(z(1+\lambda)^{1/2}) F(\alpha, \beta; \gamma; -\lambda) d\lambda = -\frac{1}{2} \Gamma(\gamma) \\ \times \frac{\Gamma(\alpha-\gamma+\nu)\Gamma(\beta-\gamma+\nu)}{\Gamma(\alpha+\beta-\gamma+\nu)} \left(\frac{z}{2}\right)^{-\nu} \\ {}_1F_2\left(\begin{matrix} -\nu-\alpha-\beta+\gamma+1; -z^2/4 \\ 1-\nu-\alpha+\gamma, 1-\nu-\beta+\gamma \end{matrix}\right) + \frac{1}{2} \Gamma(\gamma) \\ \times \sum_{\alpha, \beta} \frac{\Gamma(-\nu-\alpha+\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)(z/2)^{2\gamma-\nu-2\alpha}} \cdot e^{-i\pi(\alpha-\gamma+\nu)} \\ {}_1F_2\left(\begin{matrix} 1-\beta; -z^2/4 \\ 1+\alpha-\gamma+\nu, 1+\alpha-\beta \end{matrix}\right).$$

Similarly by increasing $\arg z$ by $\pi/2$ in (10) and proceeding as before, we have if $R(\gamma) > 0$

$$(17) \quad i^{2\nu} \int_0^\infty \lambda^{\gamma-1}(1+\lambda)^{-\nu/2} G_\nu(e^{i\pi}z(1+\lambda)^{1/2}) F(\alpha, \beta; \gamma; -\lambda) d\lambda \\ = -\frac{1}{2} \Gamma(\gamma) \frac{\Gamma(\alpha-\gamma+\nu)\Gamma(\beta-\gamma+\nu)}{\Gamma(\alpha+\beta-\gamma+\nu)} \left(\frac{z}{2}\right)^{-\nu} \\ {}_1F_2\left(\begin{matrix} -\nu-\alpha-\beta+\gamma+1; -z^2/4 \\ 1-\nu-\alpha+\gamma, 1-\nu-\beta+\gamma \end{matrix}\right) \\ + \frac{1}{2} \Gamma(\gamma) \sum_{\alpha, \beta} \frac{\Gamma(-\nu-\alpha+\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)(z/2)^{2\gamma-\nu-2\alpha}} e^{i\pi(\alpha-\gamma+\nu)} \\ {}_1F_2\left(\begin{matrix} 1-\beta; -z^2/4 \\ 1+\alpha-\gamma+\nu, 1+\alpha-\beta \end{matrix}\right).$$

Hence, on applying the formula

$$\pi i J_\nu(t) = G_\nu(t) - i^{2\nu} G_\nu(e^{i\pi}t),$$

and (15); formula (14) is obtained.

PARTICULAR CASES. In (14), take $\alpha = -m-n$, $\beta = -m+n+1$, $\gamma = 1-m$ and apply (3), so getting formula (2). A particular case of interest is obtained by putting $\alpha = 0$ in (14). Thus we have if $R(\gamma) > 0$, $R(\nu/2-1/4) > 0$, $R(z) > 0$,

$$(18) \quad \int_0^\infty \frac{J_\nu(z(1+\lambda)^{1/2})}{(1+\lambda)^{\nu/2}} \lambda^{\gamma-1} d\lambda = \Gamma(\gamma) (2/z)^\gamma J_{\nu-\gamma}(z),$$

which is again in Watson's book p. 417, formula (5).

INSTITUTE FOR ADVANCED STUDY AND
EIN SHAMS UNIVERSITY, CAIRO, EGYPT