CHEBYCHEV APPROXIMATION IN LOCALLY COMPACT SPACES

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It is well known that if f is a real continuous function on the unit interval I , then the best polynomial p of degree m , in the Chebychev sense, that approximates f has the property that if e is the error function, $e = f - p$, then there exist $m+2$ points $x_0 < x_1 < \cdots < x_{m+1}$ in I such that $(-1)^i e(x_i) = u \|e\|$, where $u = \pm 1$, and

$$\|e\| = \max \{ |e(x)| : x \in I \}.$$

Moreover, this p is unique, and is characterized by this property.

In this paper we shall give a generalization of this result (except for uniqueness, which does not hold) on a locally compact space X .

Let $B = C(X)$ be the Banach space of all continuous real functions on X , vanishing at infinity, with $\|f\| = \max \{ |f(x)| : x \in X \}$. Let $g_1, \cdots, g_n$ be n fixed functions in B , linearly independent, and let $P = V\{g_1, \cdots, g_n\}$. The Chebychev approximation problem is to find p_0 in P such that $\|f - p_0\| \leq \|f - p\|$ for all p in P . We can then assert:

THEOREM. *Let $f \in B$, $p_0 \in P$, $e_0 = f - p_0$. Then n.a.s.c. that $\|f - p_0\| \leq \|f - p\|$ for all p in P is that for some $r \leq n$, there exist $r+1$ points $x_0, \cdots, x_r$ in X such that the $n \times (r+1)$ matrix $\{g_i(x_j)\}$, $i = 1, 2, \cdots, n$; $j = 0, 1, \cdots, r$ has rank r , and such that if the first r rows are independent (renumber the g_i if necessary), and a_i is the sgn of the cofactor of a_i in the matrix*

$$\begin{pmatrix} a_0 & \cdots & a_r \\ g_1(x_0) & \cdots & g_1(x_r) \\ \vdots & & \vdots \\ g_r(x_0) & \cdots & g_r(x_r) \end{pmatrix},$$

then $e_0(x_j) = a_j \|e_0\| u$ for all j such that $a_j \neq 0$, where $u = 1$ or $u = -1$.

In general, p_0 is not unique. Also for a given p_0 satisfying the conditions of the theorem, the points $x_0, \cdots, x_r$ are not unique, and not even the integer r is unique.

Finally, we remark that there is at least one important special case where the integer r of the theorem can have only the value

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$r = n$. This is the case where X is a compact subset of the real line and $g_i(x) = x^{i-1}$, $i = 1, 2, \dots, n$. It is this case that gives the classical theorem stated at the beginning (except for uniqueness, which must be proved separately).

To prove our theorem, we go to the dual space B^* of B . We know that B^* is the space of all regular finite signed measures on X ; the measurable sets are the Borel sets of X . For every F in B^* , we have the Jordan decomposition $F = F_1 - F_2$, where F_1 and F_2 are (positive) measures on disjoint subsets. If we put $G = F_1 + F_2$, then $\|F\| = \int_X dG$.

Throughout the following, we shall let $N = \{F \in B^*: \int p dF = 0 \text{ for all } p \text{ in } P\}$, and S denotes the unit sphere of B^* . We have $(B/P)^* \cong N$, an algebraic isomorphism and an isometry, so that $\inf\{\|f - p\|: p \in P\} = \|\|f + P\|\| = \sup\{\int f dF: F \in N \cap S\}$, where $\|\|f + P\|\|$ is the norm in B/P .

Since P is finite dimensional, there is certainly some p_0 in P such that $\|f - p_0\| = \inf\{\|f - p\|: p \in P\}$. We let $e_0 = f - p_0$. Since $N \cap S$ is W^* -compact, there is some F_0 in $N \cap S$ such that $\int f dF_0 = \sup\{\int f dF: F \in N \cap S\}$. Then $\int e_0 dF_0 = \int (f - p_0) dF_0 = \int f dF_0 = \|f - p_0\| = \|e_0\|$. We let $C = \{F \in N \cap S: \int e_0 dF = \|e_0\|\}$. Then $C \neq \emptyset$. Since C is convex and W^* -compact, C is generated by its extreme points.

For every F in B^* , we let $\Lambda(F)$ denote the support of F .

LEMMA 1. *Let F_0 be an extreme point of C . Let P_0 denote the space of functions of P , restricted to $\Lambda(F_0)$, and let m be the dimension of P_0 , so $m \leq n$. Then $\Lambda(F_0)$ contains exactly $m + 1$ points. Moreover, $|e_0(x)| = \|e_0\|$ for every x in $\Lambda(F_0)$.*

PROOF. The last assertion follows immediately from the facts that $\int e_0 dF_0 = \|e_0\|$ and $\|F_0\| = 1$.

Let $F_0 = F_1 - F_2$ denote the Jordan decomposition of F_0 , and set $G = F_1 + F_2$. Let $v = e_0 / \|e_0\|$. Then $v(x) = 1$ on $\Lambda(F_1)$, $v(x) = -1$ on $\Lambda(F_2)$, and $\int g dF_0 = \int g v dG$ for every g in B .

Let $H = L_2(G)$. Then P_0 has dimension m in H , and $\int p dF_0 = \int p v dG = 0$ for all p in P_0 , so that H is at least $m + 1$ dimensional. Our desired conclusion will follow if we show that H is exactly $m + 1$ dimensional.

If $\dim H > m + 1$, choose s in B such that $s \perp \bigvee\{P, v\}$, $s \neq 0$. This is possible, for B being dense in H , choose any s_1 in B , not in $\bigvee\{P, v\}$, and write $s_1 = s_2 + s$ with s_2 in $\bigvee\{P, v\}$, $s \perp \bigvee\{P, v\}$.

Since s and $v \in B$, hence are bounded, there is a positive c such that $1 \pm cs v > 0$ on X , and since $|v(x)| = 1$ a.e. $[G]$, we have $\int |v \pm cs| dG = \int |1 \pm cs v| dG = \int [1 \pm cs v] dG = \int dG = 1$, because $s \perp v$. We now define F_3 and F_4 in B^* by $dF_3 = (v + cs)dG$, and $dF_4 = (v - cs)dG$. Then F_3 and $F_4 \in N \cap S$, clearly. Also, F_3 and $F_4 \in C$ because, for example,

$\int e_0 dF_3 = \|e_0\| \int v(v+cs) dG = \|e_0\| \int v^2 dG = \|e_0\|$. Finally, $F_0 = (F_3 + F_4)/2$, and we have reached our contradiction, because F_0 is an extreme point of C .

We are now ready to prove our theorem.

Suppose, then, that p_0 satisfies $\|f - p_0\| \leq \|f - p\|$ for every p in P ; $p_0 \in P$. Let F_0 be an extreme point of C . Then $\Lambda(F_0)$ contains a finite number $m+1$ points, $m \leq n$. For the integer r of our theorem, we take $r = m$, and take $x_0, \dots, x_r$ to be the points of $\Lambda(F_0)$. The $n \times (r+1)$ matrix $\{g_i(x_j)\}$, $i=1, 2, \dots, n$; $j=0, 1, \dots, r$, must have rank r , since its row space is isomorphic to P_0 , of dimension $m=r$. By renumbering the g_i , we may suppose that the first r rows of our matrix are independent.

Now our measure F_0 is of the form $\int h dF_0 = \sum_{j=0}^r h(x_j) m_j$ for every h in B . Since $F_0 \in N$, the m_j must satisfy

$$\begin{pmatrix} g_1(x_0) & \cdots & g_1(x_r) \\ \vdots & & \vdots \\ g_r(x_0) & \cdots & g_r(x_r) \end{pmatrix} \begin{pmatrix} m_0 \\ \vdots \\ m_r \end{pmatrix} = 0.$$

The coefficient matrix having rank r implies that the m_j are given uniquely apart from a scalar factor by the cofactors of a_j in the matrix

$$T = \begin{pmatrix} a_0 & \cdots & a_r \\ g_1(x_0) & \cdots & g_1(x_r) \\ \vdots & & \vdots \\ g_r(x_0) & \cdots & g_r(x_r) \end{pmatrix},$$

and the condition $\|F_0\|=1$, or $\sum_0^r |m_j|=1$, now fixes the m_j apart from a factor ± 1 . No m_j can be zero here, or else $\Lambda(F_0)$ would contain fewer than $r+1$ points. Finally, the equation $\|e_0\| = \int e_0 dF_0 = \sum_0^r e_0(x_j) m_j$ together with $\sum_0^r |m_j|=1$ imply that $\text{sgn } m_j = \text{sgn } e_0(x_j)$, $j=0, \dots, r$, and if we set a_j equal to the sgn of its cofactor in the matrix T , we have $a_j = \text{sgn } m_j$, $j=0, \dots, r$, or else $a_j = -\text{sgn } m_j$, $j=0, \dots, r$, and we conclude that $e_0(x_j) = u a_j \|e_0\|$, $u=1$ or $u=-1$, and our theorem is half proved.

Conversely, if for some $r \leq n$, the $n \times (r+1)$ matrix $\{g_i(x_j)\}$, $i=1, \dots, n$; $j=0, 1, \dots, r$ has rank r for some $r+1$ distinct points $x_0, \dots, x_r$, and if a_j is the sgn of its cofactor in the matrix T above, supposing the first r rows of our $n \times (r+1)$ matrix are independent, and if, furthermore, we have $e_0(x_j) = a_j u \|e_0\|$ for all j such that $a_j \neq 0$, $j=0, \dots, r$, with $u = \pm 1$, then we assert $\|f - p_0\| \leq \|f - p\|$ for all p in P .

In fact, if $\|f - q\| < \|f - p_0\| = \|e_0\|$ for some q in P , then for every F in $N \cap S$, we have $\int f dF = \int (f - q) dF \leq \|f - q\| < \|e_0\|$. On the other hand, we can construct an F_0 in $N \cap S$ for which $\int f dF_0 = \|e_0\|$. Indeed, we need only solve the system

$$\begin{pmatrix} g_1(x_0) & \cdots & g_1(x_r) \\ \vdots & & \vdots \\ g_r(x_0) & \cdots & g_r(x_r) \end{pmatrix} \begin{pmatrix} m_0 \\ \vdots \\ m_r \end{pmatrix} = 0$$

with $\sum_0^r |m_j| = 1$. The coefficient matrix has rank r , so the solution m_j is given, apart from a scalar, by the cofactor of a_j in T , above, and not all the m_j are zero. Since $\sum_0^r |m_j| = 1$, the measure F_0 , given by $\int h dF_0 = \sum_0^r h(x_j) m_j$, is in $N \cap S$. We now have

$$\int f dF_0 = \int (f - p_0) dF_0 = \int e_0 dF_0 = \sum_{j=0}^r e_0(x_j) m_j = u \|e_0\| \sum_0^r a_j m_j,$$

and since $\sum_0^r a_j m_j = \pm 1$, we have $\int f dF_0 = \pm \|e_0\|$; if $\int f dF_0 = -\|e_0\|$, take $-F_0$ instead of F_0 . The proof of the theorem is now complete.

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