ON THE CONVERGENCE AND SUMMABILITY OF A SERIES ASSOCIATED WITH THE DERIVED FOURIER SERIES

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1. Let $f(\theta)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π .
Let

$$(1.1) \quad \begin{aligned} f(\theta) &\sim \frac{1}{2} a_0 + \sum_1^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \\ &= \frac{1}{2} a_0 + \sum_1^{\infty} A_n(\theta). \end{aligned}$$

The series conjugate to (1.1) is

$$(1.2) \quad \sum_1^{\infty} (b_n \cos n\theta - a_n \sin n\theta) = \sum_1^{\infty} B_n(\theta)$$

and the differentiated series of (1.1) at $\theta = x$ is

$$(1.3) \quad \sum_1^{\infty} n B_n(x).$$

Let

$$(1.4) \quad \psi(t) = f(x+t) - f(x-t), \quad g(t) = \frac{\psi(t)}{4 \sin t/2}.$$

Throughout the present paper we assume $g(t)$ to be $L(0, \pi)$ and defined outside $(-\pi, \pi)$ by periodicity. In this paper we have considered the question of convergence and summability of the series

$$(1.5) \quad \sum_1^{\infty} \frac{S_n(x)}{n},$$

where

$$(1.6) \quad S_n(x) = \sum_{r=1}^n r B_r(x).$$

In the third section of the paper we have proved that the series (1.5) behaves more or less like the conjugate series (1.2) so far as the convergence is concerned, the function g in the present problem play-

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ing the role of ψ in the corresponding one for the conjugate series. It is possible to prove this by Tauberian argument from known results in the theory of Fourier Series, but the direct proof given in this paper seems to be interesting. We have proved a test for the convergence of (1.5) analogous to Pollard's generalization of Lebesgue's test for the convergence of Fourier Series [8].¹ More precisely we have proved the following

THEOREM 1. *If*

$$(1.7) \quad \lim_{k \rightarrow \infty} \limsup_{v \rightarrow 0} \int_{kv}^{\eta} \left| \frac{g(t+v)}{t+v} - \frac{g(t)}{t} \right| dt = 0 \quad (0 < \eta \leq \pi)$$

and

$$(1.8) \quad \int_{-0}^{\pi} g(t) \csc \frac{1}{2} t dt$$

exists as a Cauchy integral down to the origin, then the series (1.5) is convergent.

The problem of Cesàro summability of order ρ (≥ 1) has been considered for the series (1.5) in the fourth section. Here we have observed that like convergence the Cesàro summability of order ρ (> 1) of the series (1.5) is also analogous to that of the conjugate series.

Finally, in the last section, we have established a set of sufficient conditions for the logarithmic summability of order one of the series (1.5).

2. The following known lemmas are used in the proof of Theorem 1.

LEMMA 1. *Let*

$$(2.1) \quad h(n, t) = \sum_{\mu=1}^n \frac{\sin \mu t}{\mu}.$$

If

$$(2.2) \quad \int_0^t |g(u)| du = o(t),$$

then a necessary and sufficient condition that

$$(2.3) \quad \int_0^{\pi} g(t) \csc \frac{1}{2} t h(n, t) dt$$

tend to a limit as $n \rightarrow \infty$ is the existence of the integral (1.8).

¹ The numbers within square brackets refer to the bibliography at the end.

LEMMA 2. *The condition (1.7) implies that*

$$(2.4) \quad \int_0^t |g(u)| du = O(t).$$

For the proof of Lemma 1, see [10] and [5]. Lemma 2 is a known result due to Gergen [2].

3. PROOF OF THEOREM 1. Let $T_n(x)$ denote the n th partial sum of the series (1.5), so that

$$(3.1) \quad T_n(x) = \sum_{r=1}^n \frac{S_r(x)}{r}.$$

Now

$$\begin{aligned} S_r(x) &= \sum_{m=1}^r m B_m(x) \\ &= \frac{1}{\pi} \int_0^\pi \psi(t) \sum_{m=1}^r m \sin mtdt \\ &= -\frac{1}{\pi} \int_0^\pi \psi(t) \frac{d}{dt} \left(\frac{1}{2} + \sum_{m=1}^r \cos mt \right) dt \\ &= -\frac{1}{\pi} \int_0^\pi \psi(t) \frac{d}{dt} \left\{ \frac{\sin(r+1/2)t}{2 \sin t/2} \right\} dt. \end{aligned}$$

Hence,

$$\begin{aligned} T_n(x) &= -\frac{1}{\pi} \int_0^\pi \psi(t) \frac{d}{dt} \left\{ \frac{1}{2} \csc \frac{1}{2} t \sum_{r=1}^n \frac{\sin(r+1/2)t}{r} \right\} dt \\ &= -\frac{1}{\pi} \int_0^\pi g(t) \csc \frac{1}{2} t \sin(n+1)t dt \\ (3.2) \quad &+ \frac{1}{\pi} \int_0^\pi g(t) \csc \frac{1}{2} t h(n, t) dt + \frac{2}{\pi} \int_0^\pi g(t) \cos \frac{1}{2} t dt \\ &= -I_1 + I_2 + I_3, \text{ say.} \end{aligned}$$

I_3 is a constant.

Now the existence of (1.8) as a Cauchy integral implies that [7]

$$(3.3) \quad \int_0^t g(u) du = o(t),$$

which together with (1.7) shows that $I_1 \rightarrow 0$ as $n \rightarrow \infty$, by Pollard-

Lebesgue test for the convergence of Fourier Series [8]. Finally

$$\begin{aligned} I_2 &= \frac{1}{\pi} \int_0^\pi g(t) \csc \frac{1}{2} t h(n, t) \\ &= \frac{2}{\pi} \int_0^\pi \frac{g(t)}{t} h(n, t) dt + \frac{1}{\pi} \int_0^\pi \left(\csc \frac{1}{2} t - \frac{2}{t} \right) g(t) h(n, t) dt \\ &= I_{2,1} + I_{2,2}, \text{ say.} \end{aligned}$$

Evidently $I_{2,2}$ tends to a limit as $n \rightarrow \infty$ and by partial integration

$$(3.4) \quad I_{2,1} = -\frac{2}{\pi} \int_0^\pi \left(\int_0^t g(u) du \right) \frac{d}{dt} \left(\frac{h(n, t)}{t} \right) dt,$$

the integrated terms vanishing on account of (3.3). Defining

$$(3.5) \quad g_1(t) = \frac{1}{t} \int_0^t g(u) du,$$

we have

$$\begin{aligned} I_{2,1} &= \frac{2}{\pi} \int_0^\pi \frac{g_1(t)}{t} h(n, t) dt \\ &\quad - \frac{2}{\pi} \int_0^\pi g_1(t) \frac{\sin(n + 1/2)t}{2 \sin t/2} dt \\ &\quad + \frac{2}{\pi} \int_0^\pi g_1(t) dt \\ &= J_1 - J_2 + J_3, \end{aligned}$$

suppose. As J_3 is a finite constant we need only consider J_1 and J_2 . We have

$$(3.6) \quad \lim_{t \rightarrow +0} g_1(t) = 0$$

and thus

$$(3.7) \quad g_1(t) \text{ is } L(0, \pi).$$

Besides

$$(3.8) \quad \int_0^t |d\{ug_1(u)\}| = O(t),$$

since the left hand side of (3.7)

$$\begin{aligned}
&= \int_0^t \left| d \left\{ \int_0^u g(v) dv \right\} \right| \\
&= \int_0^t |g(u)| du \\
&= O(t) \text{ by Lemma 2.}
\end{aligned}$$

Thus all the conditions of Young's test for the convergence of Fourier Series [9] are satisfied and therefore

$$J_2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now

$$\begin{aligned}
(3.9) \quad J_1 &= \frac{1}{\pi} \int_0^\pi g_1(t) \csc \frac{1}{2} t h(n, t) dt \\
&\quad - \frac{1}{\pi} \int_0^\pi g_1(t) \left(\csc \frac{1}{2} t - \frac{2}{t} \right) h(n, t) dt = J_{1,1} - J_{1,2}, \text{ say.}
\end{aligned}$$

Obviously $J_{1,2}$ tends to a finite limit. Also $J_{1,1}$ tends to a limit by Lemma 1, since the existence of

$$\int_0^\pi g_1(t) \csc \frac{1}{2} t dt$$

follows from the existence of the integral (1.8), by partial integration. This completes the proof of Theorem 1.

4. If we assume merely the existence of (1.8) as a Cauchy integral, it is not difficult to show that the series (1.5) is summable (c, ρ) where $\rho > 1$. For, the existence of (1.8) as a Cauchy integral implies (3.3) and therefore, by a result due to Bosanquet [1], the sequence represented by I_1 of (3.2) is summable (c, ρ) where $\rho > 1$. The existence of Cauchy integral (1.8) is known to be a sufficient condition for the $(C, 1)$ summability of the sequence represented by I_2 of (3.2). This result is due to Hardy and Littlewood [5]. Hence from (3.2) the required result follows immediately. Thus the problem of summability (c, ρ) , $\rho > 1$ of (1.5) is more or less analogous to that of the conjugate series (1.2).

Incidentally, if (1.8) exists as a Cauchy integral whenever

$$\int_0^t |g(u)| du = O(t),$$

then the series (1.5) is summable $(C, 1)$. For, we have proved in the

preceding sections that the sequence I_2 of (3.2) is convergent under the present condition and I_1 is obviously summable $(C, 1)$ by a result due to Hardy and Littlewood [6].

5. With regard to the logarithmic summability of order one of the series (1.5), one has the following

THEOREM 2. *If as $t \rightarrow +0$*

$$(5.1) \quad \int_0^t |g(u)| du = o\left(t \log \frac{1}{t}\right),$$

$$(5.2) \quad G(t) = \int_t^\pi g(u) \csc \frac{1}{2} u du = o\left(\log \frac{1}{t}\right)$$

and

$$(5.3) \quad \int_t^\pi \frac{G(u)}{u} du = o\left(\log \frac{1}{t}\right)$$

then the series (1.5) is summable $(R, 1)$.²

The proof of Theorem 2 rests on the following

LEMMA. *If (5.1) holds (a condition satisfied whenever $g(t) = o(\log(1/t))$ and in particular when $g(t)$ is bounded near $t = x$) a necessary and sufficient condition that the sequence*

$$(5.4) \quad \left\{ \int_0^\pi g(t) \csc \frac{1}{2} t \sin \left(n + \frac{1}{2} \right) t dt \right\}$$

may be summable $(R, 1)$ is that

$$(5.5) \quad \int_t^\pi \frac{g(u)}{u} du = o\left(\log \frac{1}{t}\right).$$

This lemma is a re-statement of a theorem due to Hardy on the Riesz logarithmic summability of Fourier Series [4].

For the proof of Theorem 2 the parts I_1 and I_2 of the sequence $\{T_n(x)\}$ given by (3.2) have to be considered, in so far as I_3 is a constant. Now

$$I_1 = \frac{1}{\pi} \int_0^\pi g(t) \csc \frac{1}{2} t \sin \left(n + \frac{1}{2} \right) t dt + o(1).$$

² The sequence $\{S_n\}$ is said to be summable $(R, 1)$ to S if $\sigma_n = (S_1 + S_2 + \dots + S_n)/n \rightarrow S$ as $n \rightarrow \infty$. Obviously a series $\sum u_n$ is summable $(R, 1)$ if the sequence representing its n th partial sum is thus summable.

Therefore, since (5.2) implies (5.5), I_1 is summable $(R, 1)$ by the above Lemma.

Finally, differentiating (5.2) with respect to t we have

$$G'(t) = -g(t) \csc t/2$$

and therefore,

$$\begin{aligned} I_2 &= -\frac{1}{\pi} \int_0^\pi G'(t) h(n, t) dt \\ (5.6) \quad &= -\frac{1}{\pi} \left[G(t) h(n, t) \right]_0^\pi + \frac{1}{\pi} \int_0^\pi G(t) h'(n, t) dt \\ &= \frac{1}{2\pi} \int_0^\pi G(t) \csc \frac{1}{2} t \sin \left(n + \frac{1}{2} \right) t dt - \frac{1}{2\pi} \int_0^\pi G(t) dt \\ &= Q_1 + Q_2 \text{ say.} \end{aligned}$$

The integrated terms in (5.6) vanish, because for the lower limit

$$G(t) h(n, t) = \left\{ o \left(\log \frac{1}{t} \right) O(n, t) \right\} = o \left(nt \log \frac{1}{t} \right) = o(1)$$

for all n as $t \rightarrow +0$.

Now $G(t)$ is known [3] to be an even function and integrable $L(0, \pi)$. Hence Q_2 is a constant, and since

$$(5.7) \quad \int_0^t |G(u)| du = o \left(t \log \frac{1}{t} \right)$$

follows from (5.2), Q_1 is summable $(R, 1)$ by the Lemma with $g(t)$ replaced by $G(t)$. This completes the proof of Theorem 2.

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