A CONTINUOUS EXACT SET¹

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A linearly ordered set $\mathfrak{A}$ is called *exact*² if it is not similar to any of its proper subsets. It is easily seen that $\mathfrak{A}$ is exact if and only if the only similarity transformation of $\mathfrak{A}$ into itself is the identity [9, p. 341]. The first example of an exact set was given by Dushnik and Miller in [5]; for related investigations, such as the decomposition of a set into exact subsets, see the bibliography below. The standard construction is to delete suitably chosen elements from a continuous set (e.g., the reals), with the result that the exact set so obtained has gaps. Cuesta [2] has asked whether there can exist a *continuous* exact set.³

In this paper, we construct a continuous exact set of power $c = 2^{\aleph_0}$. The proof of exactness, however, requires the arithmetical hypothesis that c is a *regular* cardinal number, that is, that the sum of fewer than c cardinals, each of which is less than c , is itself less than c . (In particular, this will be the case if the continuum hypothesis is true.)

1. **The family $\mathfrak{F}$.** The closed interval $[0, 1]$ of the reals will be denoted throughout by L . From results of Ginsburg, e.g., [9, Theorem 2.2], one obtains an unbordered subset E of L such that every interval of L contains c elements of E , and such that no two disjoint subsets of E of power c are similar. Let $\mathfrak{F}$ be a family of c mutually disjoint subsets of E , such that, for every $F \in \mathfrak{F}$,

(1.1) *every interval of L contains c elements of F .*

To obtain such a decomposition is not difficult.

Since distinct elements of $\mathfrak{F}$ are disjoint, we have

(1.2) *for distinct elements F, F' of $\mathfrak{F}$, no subset of F of power c is similar to any subset of F' .*

Since each $F \in \mathfrak{F}$ is unbordered, we have

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² The term is due to S. Ginsburg.

³ I am indebted to F. Bagemihl for calling this to my attention.

$$(1.3) \quad 0 \notin F, \quad 1 \notin F \quad (F \in \mathfrak{F}).$$

The following lemma is easily generalized, but we state here only what we shall need for application below.

LEMMA 1. Let $|U| < c$, and let S, F_s ($s \in S$), and F'_u ($u \in U$) be mutually disjoint subsets of elements of $\mathfrak{F}$. Then, if c is a regular cardinal, no subset of power c of the set $T = \sum_{s \in S} F_s$ is similar to the set $W = \sum_{u \in U} F'_u$.

PROOF. Let $A \subset T$ be of power c , and suppose that there exists a similarity transformation f of A to W . If there is an $s \in S$ such that $|A \cap F_s| = c$, choose any such s , and define $B = A \cap F_s$. If there is no such s , then the regularity of c implies that A meets c different sets F_s ; in this case, A has a subset of power c that is imbeddable in S , and we define B to be any such subset. Since $|U| < c$, and c is regular, there exist a $u \in U$, and a subset C of B of power c , such that $f(C) \subset F'_u$. This, however, contradicts (1.2).

2. **The set $\mathfrak{A}$.** We write $\mathfrak{F} = (F_{n,x})$, with the indices chosen inductively as follows (distinct indices denoting distinct elements of $\mathfrak{F}$):

for $n=1$, we have $x=0$;

for $n>1$, x ranges over the set $\bigcup_y F_{n-1,y}$.

(This scheme may be accomplished by first decomposing the family $\mathfrak{F} - \{F_{1,0}\}$ into $\aleph_0$ subfamilies, each of power c .) Thus if $F_{m,y}$ exists (i.e., if this indexing occurs according to the above scheme), then $F_{m+1,x}$ exists for each $x \in F_{m,y}$. For every $F \in \mathfrak{F}$, each $x \in F$ appears just once as a second index; to each first index $n>1$, there correspond c second indices x ; and $F_{n,x} = F_{m,y}$ if and only if $n=m$ and $x=y$.

The notation $x \in F_{m,y}$ will be understood to include the case that $F_{m,y}$ does not exist. With this convention, we define $\mathfrak{A}$ to be the set of all sequences $\xi = (x_i)_{0 < i < \omega}$ of elements of L , with coordinates x_i restricted inductively as follows. Define $x_0 = 0$. Then

$$(2.1) \quad \begin{aligned} &(x_i) \in \mathfrak{A} \text{ if and only if } x_1 \in L, \text{ and for each } i > 0, \\ &\quad (a) \text{ if } x_i \in F_{i,x_{i-1}}, \text{ then } x_{i+1} \in L, \\ &\quad (b) \text{ if } x_i \notin F_{i,x_{i-1}}, \text{ then } x_{i+1} = 0. \end{aligned}$$

The set $\mathfrak{A}$ will be our continuous exact set.

From (2.1b), if $n>0$ and $x_{n+1} \neq 0$, then $x_n \in F_{n,x_{n-1}}$, whence, by the indexing scheme, F_{n+1,x_n} exists. For $n=0$, $F_{n+1,x_n} = F_{1,0}$ exists in any case. So we have:

$$(2.2) \quad \text{if } (x_i) \in \mathfrak{A}, \text{ and } x_{n+1} \neq 0, \text{ then } F_{n+1,x_n} \text{ exists} \quad (n=0, 1, \dots).$$

Since $0 \notin F$ ($F \in \mathfrak{F}$), iteration of (2.1b) yields

$$(2.3) \quad \text{if } (x_i) \in \mathfrak{A}, \text{ and } x_{n+1} \notin F_{n+1, x_n}, \text{ then } x_j = 0 \text{ for all } j > n+1 \\ (n = 0, 1, \dots).$$

$\mathfrak{A}$ has a first element, namely $(0, 0, \dots)$. Since $1 \notin F_{1,0}$, $\mathfrak{A}$ has a last element, namely, $(1, 0, 0, \dots)$; we denote this latter by π .

For each $\xi = (x_i) \in \mathfrak{A}$, we define the finite sequences

$$(2.4) \quad \begin{aligned} \phi_n(\xi) &= (x_1, \dots, x_n) & (n = 1, 2, \dots), \\ \phi_{-1}(\xi) &= \phi_0(\xi) = \emptyset. \end{aligned}$$

LEMMA 2. Let $\xi = (x_i) \in \mathfrak{A}$ and $\eta = (y_i) \in \mathfrak{A}$, and consider any fixed $n = 1, 2, \dots$. If $x_{n+1} \neq 0$, $y_{n+1} \neq 0$, and F_{n+1, x_n} meets F_{n+1, y_n} (cf. (2.2)) then $\phi_n(\xi) = \phi_n(\eta)$.

PROOF. Since distinct elements of $\mathfrak{F}$ are disjoint, we have $x_n = y_n$, whence the result holds for $n = 1$. Since $x_{n+1} \neq 0$, we have $x_n \in F_{n, x_{n-1}}$, by (2.1b), whence $x_n \neq 0$, by (1.3). Likewise, $y_n \in F_{n, y_{n-1}}$, and the result now follows by induction.

3. Proof that $\mathfrak{A}$ is continuous. Let $\mathfrak{B}$ be any nonempty subset of $\mathfrak{A}$. Define $L_0 = \{0\}$. Consider any $m < \omega$, and suppose that a subset L_i of L has been defined for all $i \leq m$, and that L_i has a last element y_i for all $i \leq m$. If L_m has a last element y_m , define

$$L_{m+1} = \{x_{m+1}: (x_i) \in \mathfrak{B}, x_i = y_i \text{ for all } i \leq m\}.$$

If L_i is thus defined for all $i < \omega$, then, clearly, $(y_i)_{0 < i < \omega}$ is the least upper bound of $\mathfrak{B}$ in $\mathfrak{A}$.

In the contrary case, let m denote the smallest integer such that L_{m+1} is not defined, i.e., such that L_m has no last element. Denote the least upper bound of L_m in L by z_m . Define $z_i = y_i$ for all $i < m$, and $z_i = 0$ for all $i > m$. Then, evidently, $(z_i)_{0 < i < \omega}$ is the least upper bound of $\mathfrak{B}$ in $\mathfrak{A}$.

Thus every nonempty subset of $\mathfrak{A}$ has a least upper bound in $\mathfrak{A}$. Obviously, $\mathfrak{A}$ is dense. Hence $\mathfrak{A}$ is continuous.

4. The set $\mathfrak{I}$. The remainder of our paper is devoted to the proof that $\mathfrak{A}$ is exact. We suppose that, on the contrary, there exists a similarity transformation f of $\mathfrak{A}$ into $\mathfrak{A}$ that is not the identity. Let $\alpha = (a_i) \in \mathfrak{A}$ satisfy $f(\alpha) \neq \alpha$, and write $f(\alpha) = \beta = (b_i)$. We shall assume that $\alpha < \beta$, the argument in the opposite case being analogous to the one that follows. Then f maps the interval $] \alpha, \beta [$ of $\mathfrak{A}$ into the interval $] \beta, \pi [$ of $\mathfrak{A}$.

Let m denote the first index at which α and β differ: $a_i = b_i$ for all

$i < m$, and $a_m < b_m$. Then $b_m \neq 0$, so $F_{m, b_{m-1}}$ exists, by (2.2). By (1.1), every interval of L contains c elements of $F_{m, b_{m-1}}$. Hence the set

$$(4.1) \quad S =]a_m, b_m[\cup F_{m, b_{m-1}}$$

is of power c . Evidently, $F_{m+1, s}$ exists for every $s \in S$. Therefore the set

$$(4.2) \quad T = \sum_{s \in S} F_{m+1, s}$$

has c mutually disjoint intervals.

Let $\mathfrak{S}$ denote the set of all elements $\theta = (h_i) \in \mathfrak{A}$ such that $h_i = b_i$ for all $i < m$, $h_m \in S$, $h_{m+1} \in F_{m+1, h_m}$, $h_{m+2} \in L$, and $h_i = 0$ for all $i > m+2$. Then for all $\theta \in \mathfrak{S}$, we have $a_m < h_m < b_m$, whence $\alpha < \theta < \beta$. Thus, $\mathfrak{S} \subset]\alpha, \beta[$.

Since f maps $]\alpha, \beta[$ into $]\beta, \pi[$, there is a subset $\mathfrak{R}$ of $]\beta, \pi[$ that is similar to $\mathfrak{S}$. Now, clearly, $\mathfrak{S}$ is similar to LT . Consequently, we may write

$$(4.3) \quad \mathfrak{R} = \sum_{\tau \in \mathfrak{T}} \mathfrak{R}_\tau,$$

where

$$(4.4) \quad \mathfrak{R}_\tau \simeq T,$$

and $\mathfrak{R}_\tau \simeq L$ for all $\tau \in \mathfrak{T}$. Furthermore, we may suppose that τ is the first element of $\mathfrak{R}_\tau$ ($\tau \in \mathfrak{T}$), so that we have $\mathfrak{T} \subset \mathfrak{R}$. Obviously, $\beta < \kappa$ for all $\kappa \in \mathfrak{R}$; in particular, then,

$$(4.5) \quad \beta < \tau \text{ for all } \tau \in \mathfrak{T}.$$

We now define

$$(4.6) \quad \mathfrak{R}_n = \{\phi_n(\tau) : \tau \in \mathfrak{T}\} \quad (n = -1, 0, 1, 2, \dots),$$

where ϕ_n is as in (2.4) (thus, $\mathfrak{R}_{-1} = \mathfrak{R}_0 = \{\emptyset\}$); and

$$(4.7) \quad \mathfrak{T}_{n, \rho} = \{(t_n, t_{n+1}) : \tau = (t_i) \in \mathfrak{T}, \phi_{n-1}(\tau) = \rho\} \\ (\rho \in \mathfrak{R}_{n-1}, n = 0, 1, \dots).$$

Obviously, for each $\rho \in \mathfrak{R}_{n-1}$, $\mathfrak{T}_{n, \rho}$ is a subset of the set

$$(4.8) \quad \mathfrak{A}_{n, \rho} = \{(x_n, x_{n+1}) : \xi = (x_i) \in \mathfrak{A}, \phi_{n-1}(\xi) = \rho\}.$$

5. The sets T_n . Next, we define

$$(5.1) \quad T_n = \{t_n : \tau = (t_i) \in \mathfrak{T}\} \quad (n = 0, 1, \dots), \\ T_{-1} = \{\emptyset\}.$$

(Note that $T_0 = \{0\}$.) Most of the rest of the paper is concerned with the proof of the following

LEMMA. $|T_i| < c$ for all i .

The proof is by induction. Trivially, $|T_{-1}| = |T_0| < c$. Fix $n \geq 0$, and suppose that $|T_i| < c$ for all $i \leq n$. Then

$$(5.2) \quad |T_n| < c;$$

also, referring to (4.6), we see that

$$(5.3) \quad |\mathfrak{R}_{n-1}| \leq |T_{-1}| \cdot |T_0| \cdot \cdots \cdot |T_{n-1}| < c.$$

Now fix $\rho \in \mathfrak{R}_{n-1}$, and consider the set $\mathfrak{T}_{n,\rho}$ (4.7). For each $(u, v) \in \mathfrak{T}_{n,\rho}$, choose one element $\tau(u, v) = (t_i) \in \mathfrak{T}$ such that

$$(5.4) \quad \phi_{n-1}(\tau(u, v)) = \rho, \quad t_n = u, \quad t_{n+1} = v,$$

and denote the set of these elements $\tau(u, v)$ by $\mathfrak{T}'$:

$$(5.5) \quad \mathfrak{T}' = \{\tau(u, v) : (u, v) \in \mathfrak{T}_{n,\rho}\}.$$

Obviously,

$$(5.6) \quad \mathfrak{T}' \subset \mathfrak{T}, \quad \mathfrak{T}' \simeq \mathfrak{T}_{n,\rho}$$

(with the lexicographic order on the pairs $(u, v) \in \mathfrak{T}_{n,\rho}$).

Now define

$$(5.7) \quad \mathfrak{R}' = \sum_{\tau \in \mathfrak{T}'} \mathfrak{L}_\tau.$$

Then $\mathfrak{R}' \subset \mathfrak{R}$ (4.3).

Next, we decompose $\mathfrak{T}_{n,\rho}$ into the complementary subsets

$$(5.8) \quad \begin{aligned} \mathfrak{B} &= \{(u, v) \in \mathfrak{T}_{n,\rho} : v \notin F_{n+1,u}\}, \\ \mathfrak{B} &= \{(u, v) \in \mathfrak{T}_{n,\rho} : v \in F_{n+1,u}\} \end{aligned}$$

(in case $F_{n+1,u}$ does not exist, then $\mathfrak{B}$ is empty), which we proceed to investigate separately.

6. Proof that $|\mathfrak{B}| < c$. Consider any $(u, v) \in \mathfrak{B}$. If $\sigma = (s_i)$ is any element of $\mathfrak{A}$ such that $\phi_{n-1}(\sigma) = \rho$, $s_n = u$, and $s_{n+1} = v$, then, since $v \notin F_{n+1,u}$ (5.8), we have from (2.3) that $s_i = 0$ for all $i > n+1$. Hence $\tau(u, v)$ is the only such element $\sigma \in \mathfrak{A}$ (5.4). It follows that if (u, v) is a left-hand limit element of $\mathfrak{T}_{n,\rho}$ in the set $\mathfrak{A}_{n,\rho}$ (4.8), then $\tau(u, v)$ is a left-hand limit element of $\mathfrak{T}'$ (5.5)—hence of $\mathfrak{R}'$ (5.7)—in $\mathfrak{A}$. But then the set $\mathfrak{L}_{\tau(u,v)}$ (5.7) cannot exist.

Therefore (u, v) is not a left-hand limit element of $\mathfrak{T}_{n,\rho}$ in $\mathfrak{A}_{n,\rho}$. There accordingly exists an interval

$$\mathfrak{J}(u, v) =](u, v), (u', v')[$$

of $\mathfrak{X}_{n,\rho}$ that is free of elements of $\mathfrak{X}_{n,\rho}$. Now either $u < u'$, or $u = u'$ and $v < v'$. Since $u \in T_n$ (5.1), and $|T_n| < \mathfrak{c}$ (5.2), there are less than $\mathfrak{c}$ intervals $\mathfrak{J}(u, v)$ with $u < u'$. On the other hand, if $u = u'$, then $\mathfrak{J}(u, v)$ is similar to the interval $]v, v'[$ of L ; so for each $u \in T_n$, there are (at most) denumerably many intervals $\mathfrak{J}(u, v)$ with $u = u'$. Consequently, there are less than $\mathfrak{c}$ intervals $\mathfrak{J}(u, v)$ altogether. It follows that $|\mathfrak{B}| < \mathfrak{c}$.

7. Proof that $|\mathfrak{B}| < \mathfrak{c}$. Since $\mathfrak{B} \subset \mathfrak{X}_{n,\rho}$ (5.8), $\mathfrak{X}_{n,\rho} \simeq \mathfrak{X}'$ (5.6), $\mathfrak{X}' \subset \mathfrak{X}$ (5.6), and $\mathfrak{X} \simeq T$ (4.4), we see that T contains a subset that is similar to $\mathfrak{B}$. Define

$$(7.1) \quad \begin{aligned} U &= \{u \in T_n : \text{there exists } v \text{ such that } (u, v) \in \mathfrak{B}\}, \\ F_{u'}' &= \{v \in F_{n+1,u} : (u, v) \in \mathfrak{B}\} \quad (u \in U). \end{aligned}$$

Then $F_u' \subset F_{n+1,u}$ ($u \in U$). Referring to (5.8), we see that

$$(7.2) \quad \mathfrak{B} \simeq W, \quad \text{where} \quad W = \sum_{u \in U} F_u'.$$

Thus, T has a subset that is similar to W .

Now, according to the indexing of the family $\mathfrak{F}$, the sets F_u' ($u \in U$) are mutually disjoint. Likewise, the sets $F_{m+1,s}$ ($s \in S$) that appear in the definition of T (4.2, 4.1) are mutually disjoint, and each is disjoint from S . Furthermore, if $n < m-1$, or $n > m$, then every F_u' is disjoint from S and from every $F_{m+1,s}$. Also, every F_u' is disjoint from every $F_{m+1,s}$ in case $n = m-1$, and from S in case $n = m$.

Suppose that $n = m-1$, and that there is a $u \in U$ such that $F_{n+1,u}$ meets S . Now $S \subset F_{m,b_{m-1}}$ (4.1); hence $F_{n+1,u}$ meets $F_{m,b_{m-1}}$. Consider any $v \in F_u'$. Then for $(t_i) = \tau(u, v)$, we have $t_n = u$, and $t_{n+1} = v \neq 0$. Since $b_m > a_m$, we have $b_m \neq 0$. It follows from Lemma 2 that $\phi_n(\tau(u, v)) = \phi_n(\beta)$. Now $\beta < \tau(u, v)$ (4.5). Therefore we must have $b_{n+1} \leq t_{n+1}$, i.e., $b_m \leq v$. Since v was an arbitrary element of F_u' , we conclude that $F_u' \subset [b_m, 1]$. But $S \subset]a_m, b_m[$ (4.1). Consequently, F_u' does not meet S . It follows that for every $u \in U$, F_u' is disjoint from S .

Next, suppose that $n = m$, and that there exist $u \in U$ and $s \in S$ such that $F_{n+1,u}$ meets $F_{m+1,s}$. Consider any $v \in F_u'$. Then for $(t_i) = \tau(u, v)$, we have $t_n = u$, and $t_{n+1} = v \neq 0$. Since $s \in F_{m,b_{m-1}}$, there is a $\sigma = (s_i) \in \mathfrak{A}$ such that $\phi_m(\sigma) = (b_1, \dots, b_{m-1}, s)$, and $s_{m+1} \neq 0$ (2.1a). By Lemma 2, we have $\phi_n(\tau(u, v)) = \phi_m(\sigma)$. But then, since $\phi_m(\beta) = (b_1, \dots, b_{m-1}, b_m)$, and $s < b_m$ (4.1), we find that $\tau(u, v) < \beta$, contradicting (4.5). It follows that every F_u' must be disjoint from every $F_{m+1,s}$.

Thus, in all cases, the sets S , $F_{m+1,s}$ ($s \in S$), and F'_u ($u \in U$) are mutually disjoint. Finally, we have $|U| < c$, since $U \subset T_n$ (7.1) and $|T_n| < c$ (5.2). Since T (4.2) has a subset that is similar to W (7.2 ff.), we conclude from Lemma 1 (assuming that c is regular) that $|W|$ must be less than c . Therefore $|\mathfrak{B}| < c$ (7.2).

8. Conclusion. Thus, $|\mathfrak{B}| < c$ and $|\mathfrak{B}| < c$. Hence $|\mathfrak{T}_{n,\rho}| < c$ (5.8). Define

$$(8.1) \quad \mathfrak{T}_n = \{(t_n, t_{n+1}) : \tau = (t_i) \in \mathfrak{T}\}.$$

Then $\mathfrak{T}_n = \bigcup_{\rho \in \mathfrak{R}_{n-1}} \mathfrak{T}_{n,\rho}$ (4.7). Now $|\mathfrak{T}_{n,\rho}| < c$ for each $\rho \in \mathfrak{R}_{n-1}$, as has just been shown, and $|\mathfrak{R}_{n-1}| < c$, by (5.3). It follows (since c is regular) that $|\mathfrak{T}_n| < c$. Now from (5.1) and (8.1), it is clear that $|T_{n+1}| \leq |\mathfrak{T}_n|$. Hence $|T_{n+1}| < c$. This is our induction step, and the proof of the lemma is now complete.

Finally, define $M = \bigcup_{0 < n < \omega} T_n$. Then⁴ $|M| < c$. We may as well suppose that M contains all the rational numbers in L . Let $\mathfrak{M}$ denote the set of all $\xi = (x_i) \in \mathfrak{A}$ for which every $x_i \in M$. Obviously, $\mathfrak{T} \subset \mathfrak{M}$ (5.1). Now $\mathfrak{M}$ has a dense subset of power $|M| < c$, namely, the set of all $\xi \in \mathfrak{M}$ for which only finitely many $x_i \neq 0$. On the other hand, $\mathfrak{T} \simeq T$ (4.4), whence $\mathfrak{T}$ has c mutually disjoint intervals (4.2 ff.). This contradiction completes the proof that $\mathfrak{A}$ is an exact set.

BIBLIOGRAPHY

1. F. Bagemihl and L. Gillman, *Generalized dissimilarity of ordered sets*, Fund. Math. vol. 42 (1955) pp. 141-165.
2. N. Cuesta, *Permutaciones continuas con los numeros reales*, Revista Matemática Hispano-Americana vol. 5 (1945) pp. 191-203.
3. ———, *Ordenacion densa perfectamente escalonada*, Revista Matemática Hispano-Americana vol. 8 (1948) pp. 57-71.
4. ———, *Escalonamiento ordinal*, Revista Matemática Hispano-Americana vol. 14 (1954) pp. 237-268.
5. B. Dushnik and E. W. Miller, *Concerning similarity transformations of linearly ordered sets*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 322-326.
6. S. Ginsburg, *Some remarks on order types and decompositions of sets*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 514-535.
7. ———, *Further results on order types and decompositions of sets*, Trans. Amer. Math. Soc. vol. 77 (1954) pp. 120-150.
8. ———, *Fixed points of products and sums of simply ordered sets*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 554-565.
9. ———, *Order types and similarity transformations*, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 341-361.
10. W. Sierpiński, *Types d'ordre des ensembles linéaires*, Fund. Math. vol. 37 (1950) pp. 253-264.

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⁴ This follows from König's theorem, whether c is regular or not.