ORDER-COMPATIBLE TOPOLOGIES ON A PARTIALLY ORDERED SET

E. S. WOLK

1. **Introduction.** Let X be a partially ordered set (poset) with respect to a relation $\leqq$, and possessing least and greatest elements 0 and I respectively. There are many known ways of using the order properties of X to define an "intrinsic" topology on X. It is our purpose in this note, instead of considering certain special topologies of this type, to introduce a class of topologies on X which are compatible, in a natural sense, with its order. To this end, let us call a subset S of X *up-directed* (*down-directed*) if and only if for all $x \in S$ and $y \in S$ there exists $z \in S$ with $z \geqq x$, $z \geqq y$ ($z \leqq x$, $z \leqq y$). Also, following McShane [**3**], we shall call a subset K of X *Dedekind-closed* if and only if whenever S is an up-directed subset of K and $y = \text{l.u.b. } (S)$, or S is a down-directed subset of K and $y = \text{g.l.b. } (S)$, we have $y \in K$. We now introduce the following definition, which seems to be a natural requirement for a topology on X to be harmoniously related to its order structure.

DEFINITION. If $\mathfrak{J}$ is a topology defined on X, we shall say that $\mathfrak{J}$ is *order-compatible* with X if and only if

(i) every set closed with respect to $\mathfrak{J}$ is Dedekind-closed, and

(ii) every set of the form $\{x \in X \mid a \leqq x \leqq b\}$ is closed with respect to $\mathfrak{J}$.

The main purpose of this note is to obtain a simple sufficient condition for a poset X to possess a unique order-compatible topology. We say that two elements x and y in X are *incomparable* if and only if $x \nleqq y$ and $x \ngeqq y$. Let us call a subset S of X *diverse* if and only if $x \in S$, $y \in S$, and $x \neq y$ imply that x and y are incomparable. We define the *width* of X to be the l.u.b. of the set $\{k \mid k \text{ is the cardinal number of a diverse subset of } X\}$. We shall then prove, as our main result, that a poset of finite width possesses a unique order-compatible topology, with respect to which it is a Hausdorff topological space.

2. **Preliminary definitions and lemmas.** The reader may verify that the class of all Dedekind-closed subsets of a poset X is closed with respect to arbitrary intersections and finite unions. Hence we may define a topology $\mathfrak{D}$ on X whose closed sets are precisely the Dedekind-closed subsets of X. We let $\mathfrak{I}$ denote the well-known interval topology on X, which is obtained by taking all sets of the form $[a, b]$

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$=\{x \mid a \leqq x \leqq b\}$ as a sub-basis for the closed sets. If $\mathfrak{S}$ and $\mathfrak{T}$ are any topologies on X, we define $\mathfrak{S} \leqq \mathfrak{T}$ to mean that every $\mathfrak{S}$-closed set is $\mathfrak{T}$-closed. It is then obvious that we have

LEMMA 1. *If $\mathfrak{T}$ is any order-compatible topology on X, then $\mathfrak{g} \leqq \mathfrak{T} \leqq \mathfrak{D}$.*

LEMMA 2. *If X contains no infinite diverse set, then X is a Hausdorff space in its interval topology.*

PROOF. Suppose a and b are any distinct points of X. Then [**4**] X is a Hausdorff space in its interval topology if there is a covering of X by means of a finite number of closed intervals such that no interval contains both a and b. We consider the following cases, and produce such a covering in each instance.

Case (i). a and b are incomparable. Let S be a maximal diverse subset of X containing both a and b. Consider all intervals of the form $[0, s]$ and $[s, I]$ for $s \in S$. This is a finite set of intervals satisfying the above requirements.

Case (ii). $a<b$, but $a<x<b$ for no $x \in X$. Let S be a maximal diverse subset of X containing a, and let T be a maximal diverse set containing b. Consider the following collections of intervals:

(1) all intervals of the form $[0, s]$ for $s \in S$,

(2) all intervals of the form $[t, I]$ for $t \in T$,

(3) all intervals which may exist of the form $[s, t]$ for $s \in S$ and $t \in T$, provided that $s=a$ and $t=b$ are not both true.

The union of the above three collections of intervals satisfies our requirements.

Case (iii). $a<b$ and there exists x_0 with $a<x_0<b$. Let S be a maximal diverse subset containing x_0, T a maximal diverse subset containing b. Then the union of the following three collections of intervals satisfies our requirements:

(1) all intervals of the form $[0, s]$ for $s \in S$,

(2) all intervals of the form $[t, I]$ for $t \in T$,

(3) all intervals which may exist of the form $[s, t]$ for $s \in S$, $t \in T$.

Since the above three cases dispose of all possibilities, the proof is complete.

We shall find it convenient to consider nets of elements in X. We shall follow the terminology of Bartle [**1**] and Kelley [**2**], but give all the relevant definitions. If f is a function defined on an arbitrary up-directed poset A and with values lying in X, then we say that f is a *net on A to X*. We shall use the notation $(f(\alpha), \alpha \in A)$ for such a net. A net $(g(\beta), \beta \in B)$ is said to be a *subnet* of $(f(\alpha), \alpha \in A)$ if and only if there is a mapping $\pi: B \rightarrow A$ which satisfies

(i) $g(\beta)=f(\pi(\beta))$ for all $\beta\in B$, and

(ii) given any $\alpha_0\in A$, there exists $\beta_0\in B$ such that if $\beta\geqq\beta_0$ then $\pi(\beta)\geqq\alpha_0$.

Let us call a subset of A of the form $A_\beta=\{\alpha\in A \mid \alpha\geqq\beta\}$ a *residual* subset of A. A subset C of A will be called *cofinal in* A if and only if $\alpha\in A$ implies there exists $\gamma\in C$ with $\gamma\geqq\alpha$. If f is a net on A to X, and A_β is a residual subset of A, then the net $(f(\alpha), \alpha\in A_\beta)$ will be called a *residual subnet* of f. If C is cofinal in A, then the net $(f(\alpha), \alpha\in C)$ will be called a *cofinal subnet* of f. If $\beta\in A$, we shall write $E_f(\beta)$ (or simply $E(\beta)$, if no confusion can arise) to denote the set $\{x\in X \mid x=f(\alpha)$ for some $\alpha\geqq\beta\}$. A net f on A to X is said to be *universal* if and only if given any subset $S\subset X$ then either (i) there exists $\beta\in A$ such that $E(\beta)\subset S$, or (ii) there exists $\beta\in A$ such that $E(\beta)\subset S'$, the complement of S with respect to X. It is a well-known result [**1**; **2**] that *every net possesses a subnet which is universal.*

Now let $\mathfrak{J}$ be any topology on X. We say that a net f on A to X *converges* to an element y in X if and only if for any $\mathfrak{J}$-open set U containing y, there exists $\beta\in A$ such that $E(\beta)\subset U$. If f converges to y, we write $f(\alpha)\rightarrow y$. A subset S of X is closed with respect to $\mathfrak{J}$ if and only if whenever f is a net whose range is in S and $f(\alpha)\rightarrow y$, then $y\in S$ [**2**, p. 66].

The following notation will be useful. If $S\subset X$, we write $S^*=\{x\in X \mid x\geqq s$ for all $s\in S\}$, and $S^+=\{x\in X \mid x\leqq s$ for all $s\in S\}$. If f is a net on A to X, let P_f be the union of all sets of the form $\{E(\beta)\}^+$, for some $\beta\in A$; and let Q_f be the union of all sets of the form $\{E(\beta)\}^*$, for some $\beta\in A$. Then we say that an element y in X is *medial* for f if and only if $y\in P_f^*\cap Q_f^+$. We shall need the following lemma, which was proved by Ward [**5**, Lemma 1] using the terminology of filters.

LEMMA 3 (WARD). *If f is a net with range in X, and if f converges to y in the interval topology on X, then y is medial for f.*

3. **Main results.** Our main theorem will follow as a consequence of three more lemmas.

LEMMA 4. *Let f be a net on A to X and suppose that $f(\alpha)\rightarrow y$ in the interval topology on X. If $f(\alpha)$ is incomparable with y for all $\alpha\in A$, then there exists an infinite diverse subset of X contained in the range of f.*

PROOF. Let $(u(\alpha), \alpha\in D)$ be a universal subnet of f. Since every subnet of a convergent net is convergent, and to the same limit, we have $u(\alpha) \rightarrow y$ in the interval topology on X. By Lemma 3, y is medial for u.

We shall construct inductively an infinite diverse subset of X. Select $\delta_1 \in D$ arbitrarily. Since $y \in P_u^*$ and $u(\delta_1)$ is incomparable with y, we must have $u(\delta_1) \notin P_u$. Hence the set $K_1 = \{x \in X \mid x \geqq u(\delta_1)\}$ contains no $E_u(\alpha)$ for any $\alpha \in D$. Since u is a universal net, there exists some $\alpha_1 \in D$ such that $\alpha_1 > \delta_1$ and $E_u(\alpha_1) \subset K_1' = \{x \in X \mid x \ngeqq u(\delta_1)\}$. Also, since $y \in Q_u^+$, we have $u(\delta_1) \notin Q_u$; and hence $L_1 = \{x \in X \mid x \leqq u(\delta_1)\}$ contains no $E_u(\alpha)$ for any $\alpha \in D$. Hence there exists some $\beta_1 \in D$ such that $\beta_1 > \delta_1$ and $E_u(\beta_1) \subset L_1' = \{x \in X \mid x \nleqq u(\delta_1)\}$. Select $\gamma_1 \in D$ such that $\gamma_1 \geqq \alpha_1$, $\gamma_1 \geqq \beta_1$. Then $E_u(\gamma_1) \subset E_u(\alpha_1) \cap E_u(\beta_1)$. It is clear from our construction that $u(\delta_1)$ is incomparable with each element of $E_u(\gamma_1)$. Now choose $\delta_2 \in D$ such that $\delta_2 \geqq \gamma_1$. In an analogous way we obtain α_2 and β_2 such that $E_u(\alpha_2) \subset \{x \in X \mid x \ngeqq u(\delta_2)\}$, $E_u(\beta_2) \subset \{x \in X \mid x \nleqq u(\delta_2)\}$, and $\alpha_2 > \delta_2$, $\beta_2 > \delta_2$. Then choose $\gamma_2 \in D$ such that $\gamma_2 \geqq \alpha_2$, $\gamma_2 \geqq \beta_2$. Then each element of $E_u(\gamma_2)$ is incomparable with both $u(\delta_1)$ and $u(\delta_2)$. Select $\delta_3 \geqq \gamma_2$. Continuing in the above manner we obtain an infinite sequence of distinct elements $u(\delta_1)$, $u(\delta_2)$, $u(\delta_3)$, $\cdots$, which form a diverse subset of X.

LEMMA 5. *Let f be a net on A to X, let S be the range of f, and suppose that y is medial for f. If $f(\alpha) < y$ for all $\alpha \in A$, then $y =$* l.u.b.(S).

PROOF. Suppose that there exists $z \in S^*$ with $z \ngeqq y$. Since $z \in \{E_f(\alpha)\}^*$ for all $\alpha \in A$, we have $z \in Q_f$. But $y \in Q_f^+$, and hence we have a contradiction.

The obvious dual formulation of the above lemma, and also that of the following one, may be left to the reader.

LEMMA 6. *Let X be a poset of finite width, and let f be a net on A with range $(f) = S \subset X$. Let y be an element of X such that y is the* l.u.b. *of the range of every subnet of f. Then there exists an up-directed set $M \subset S$ such that $y =$* l.u.b.(M).

PROOF. Let $k =$ width of X. Let us suppose that the lemma is false· We shall proceed to obtain a contradiction by constructing a diverse subset of X containing $k+1$ elements.

It is an easy consequence of Zorn's Lemma that every up-directed subset of a poset is contained in a maximal up-directed subset. Let M_1 be any maximal up-directed subset of S. By our assumption that the lemma is false, we must have $y \neq$ l.u.b.(M_1). Hence there exists no subnet of f with range contained in M_1. Therefore there exists $\alpha_1 \in A$ such that $E(\alpha_1) \subset S - M_1$. Now let us choose a maximal up-directed subset M_2 of $E(\alpha_1)$. Since by assumption there exists no subnet of $(f(\alpha), \alpha \in A_{\alpha_1})$ with range contained in M_2, then there is an $\alpha_2 \in A$ with $\alpha_2 > \alpha_1$ and $E(\alpha_2) \subset E(\alpha_1) - M_2$. Now choose M_3, a maximal up-directed subset of $E(\alpha_2)$, and continue the above process for k steps. We obtain sets $M_1, M_2, \cdots, M_k$; and $E(\alpha_1), E(\alpha_2), \cdots, E(\alpha_k)$, such that (with the agreement that $E(\alpha_0) = S$) M_i is a maximal up-directed subset of $E(\alpha_{i-1})$ and $E(\alpha_i) \subset E(\alpha_{i-1}) - M_i$, for $i = 1, 2, \cdots, k$.

Next let us note that, for each $i = 1, 2, \cdots, k$, $x \in E(\alpha_{i-1}) - M_i$ implies (i) $x \notin M_i^*$, and (ii) $x \not\leqq m$ for any $m \in M_i$. For if either (i) or (ii) failed to hold, then the set $M_i \cup \{x\}$ would be an up-directed subset of $E(\alpha_{i-1})$, thus contradicting the maximality of M_i. Thus for each $x \in E(\alpha_{i-1}) - M_i$ there exists $x_i \in M_i$ such that x and x_i are incomparable.

Now choose an arbitrary element, which we denote by x_{k+1}, of $E(\alpha_{k-1}) - M_k$. By the above paragraph, there exists $x_k \in M_k$ such that x_{k+1} and x_k are incomparable. Also, since $x_k \in E(\alpha_{k-2}) - M_{k-1}$, there exist $a_1 \in M_{k-1}$ and $a_2 \in M_{k-1}$ such that a_1 and x_k are incomparable, a_2 and x_{k+1} are incomparable. Let x_{k-1} be an element of M_{k-1} with $x_{k-1} \geqq a_1$, $x_{k-1} \geqq a_2$. Then x_{k-1} is incomparable with both x_k and x_{k+1}, so that the set $\{x_{k+1}, x_k, x_{k-1}\}$ is diverse. Continuing in this way, we select elements b_1, b_2, b_3 in M_{k-2} such that b_1 and x_{k-1}, b_2 and x_k, b_3 and x_{k+1} form incomparable pairs. Let x_{k-2} be an element of M_{k-2} with $x_{k-2} \geqq b_i$ $(i = 1, 2, 3)$. Then $\{x_{k+1}, x_k, x_{k-1}, x_{k-2}\}$ is a diverse set. It is clear that continuing the above construction leads to a diverse set $\{x_{k+1}, x_k, \cdots, x_1\}$ of $k+1$ distinct elements, contained in range (f).

We now have the following theorem.

THEOREM. *If X is a poset of finite width, then X possesses a unique order-compatible topology. Furthermore, with respect to this topology, X is a Hausdorff space.*

PROOF. In view of Lemmas 1 and 2, we need only to prove that the topologies $\mathfrak{I}$ and $\mathfrak{D}$ are equivalent on X. Let K be any Dedekind-closed subset of X; we shall show that K is $\mathfrak{I}$-closed. Let f be a net in K with $f(\alpha) \to y$ in the interval topology. We may assume that $f(\alpha) \neq y$ for all α. We shall prove that $y \in K$. By Lemma 4, there exists no subnet g of f such that each element of range (g) is incomparable with y. Hence there exists a residual subnet of f, which we take to be f itself, whose range consists of elements all of which are comparable with y. Then there exists (i) a cofinal subnet u of f such that y is an upper bound of range (u), or (ii) a cofinal subnet v of f such that y is a lower bound of range (v). Suppose that (i) holds (the other case is handled in the obvious dual manner). Since u converges to y in the interval topology, y is medial for u (Lemma 3). Let $S=\text{range }(u)$. By Lemma 5, $y=\text{l.u.b.}(S)$. Since every subnet of u converges to y in the interval topology, Lemma 6 now applies; and we conclude that there exists an up-directed set $M\subset S\subset K$ such that $y=\text{l.u.b.}(M)$. Since K was assumed to be Dedekind-closed, we have $y\in K$, completing the proof.

It is natural to ask whether, in the above theorem, the hypothesis that X is of finite width can be replaced by the weaker condition that X contains no infinite diverse subset. However, we have not been able to settle this question (not even in the special case when X is assumed to be a lattice).

References

1. R. G. Bartle, *Nets and filters in topology*, Amer. Math. Monthly vol. 62 (1955) pp. 551–557.

2. J. L. Kelley, *General topology*, Van Nostrand, 1955.

3. E. J. McShane, *Order-preserving maps and integration processes*, Annals of Mathematics Studies vol. 31, Princeton, 1953.

4. E. S. Northam, *The interval topology of a lattice*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 825–827.

5. A. J. Ward, *On relations between certain intrinsic topologies in partially ordered sets*, Proc. Cambridge Philos. Soc. vol. 51 (1955) pp. 254–261.

University of Connecticut