ON TORSION-FREE ABELIAN GROUPS AND LIE ALGEBRAS

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It is known that many of the classes of simple Lie algebras of prime characteristic of nonclassical type have simple infinite-dimensional analogues of characteristic zero (see, for example, [4, p. 518]). We consider here analogues of those algebras which are defined by a modification of the definition of a group algebra. Thus we consider analogues of the Zassenhaus algebras as generalized by Albert and Frank in [1].

The algebras considered are defined as follows. Let G be a nonzero abelian group, F a field, g an additive mapping from G to F, and f an alternate biadditive mapping from $G \times G$ to F. We index a basis of an algebra over F by G, denoting by u_{α} the basis element corresponding to α in G, and define multiplication by

(1)
$$u_{\alpha}u_{\beta} = \{f(\alpha, \beta) + g(\alpha - \beta)\}u_{\alpha+\beta}.$$

We designate this algebra by L(G, g, f). We shall determine necessary and sufficient conditions on f and g for the algebra L(G, g, f) to be a simple Lie algebra. We shall then consider the case in which L(G, g, f)is a simple Lie algebra of characteristic zero. This will be seen to imply that G is torsion-free. The derivations and locally algebraic derivations of L(G, g, f) will be determined in this case. Using these, we shall show that any one of these simple Lie algebras L(G, g, f)of characteristic zero determines the group G up to isomorphism and determines the mappings g and f up to a scalar multiple.

Our proof of the simplicity of L(G, g, f) and determination of the derivations of L(G, g, f) are also valid when F has prime characteristic p and G is an elementary abelian p-group. However in that case our method for showing that L(G, g, f) essentially determines G, g and f cannot be used—indeed, Ree showed in [4] that all Zassenhaus algebras of dimension p^n over F are isomorphic.

When the torsion-free abelian group G has rank one, the simple algebra L(G, g, f) over F, of characteristic zero, is isomorphic to the algebra of derivations of the group algebra of G over F. Thus the group algebra of a torsion-free abelian group of rank one determines the group. However this is a special case of a result that follows from Higman's determination of the units of group algebras in [2].

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2. Simplicity. It was shown by Albert and Frank in [1, p. 132] that when $g(\alpha) = 0$ for all α in G then the algebra L(G, g, f) is a Lie algebra and contains the one-dimensional ideal spanned by u_0 , and that when g does not vanish identically then L(G, g, f) is a Lie algebra if and only if there is an (additive) mapping h on G to F such that

(2)
$$f(\alpha, \beta) = g(\alpha)h(\beta) - g(\beta)h(\alpha)$$

for all α and β in G.

Now suppose that g does not vanish identically and that an h is given such that (2) holds. We seek to determine conditions under which L(G, g, f) will be simple. If $g(\gamma) = h(\gamma) = 0$ for some nonzero γ then in case γ has finite order q, all elements of the form

$$\sum_{i=0}^{q-1} u_{\alpha+i\gamma}$$

span a proper ideal, while if γ has infinite order, all elements of the form $u_{\alpha}+u_{\alpha+\gamma}$ span a proper ideal. Also if there is an element δ in G such that $g(\delta) = 0$ and $h(\delta) = 2$ (in particular for characteristic two, if $\delta = 0$) then all u_{α} with $\alpha \neq -\delta$ span a proper ideal of L(G, g, f).

We therefore suppose that there is no nonzero δ in G such that $g(\delta) = 0$ and $h(\delta) = 0$ or 2, and that the characteristic is not two. Thus there is no δ in G such that $g(\delta) = 0$ and $h(\delta) = \pm 1$. Also, by (2), either the kernel of g is zero or f is nonsingular.

The length $\lambda(x)$ of an element $x = \sum_{\beta} a_{\beta} u_{\beta}$ of L(G, g, f) denotes the number of nonzero coefficients a_{β} in x, and β is said to be *x*-admissible if $a_{\beta} \neq 0$.

Now, under the above conditions, let M be a nonzero ideal of L(G, g, f), x a nonzero element of M of minimal length, and suppose $\lambda(x) > 1$. If $g(\alpha) = 0$ for some x-admissible α , then with γ such that $g(\gamma) \neq 0$, we have $\lambda(xu_{\gamma}) \leq \lambda(x), f(\alpha, \gamma) + g(\alpha - \gamma) = -g(\gamma) [h(\alpha) + 1] \neq 0$ and $\alpha + \gamma$ is (xu_{γ}) -admissible. Hence we may assume that $g(\beta) \neq 0$ for some x-admissible β . Now if $g(\alpha) = 0$ for some x-admissible α then $xu_{\alpha} \neq 0$ since $f(\beta, \alpha) + g(\beta - \alpha) = g(\beta) [h(\alpha) + 1] \neq 0$, but $\lambda(xu_{\alpha}) < \lambda(x)$, a contradiction. Hence for every x-admissible β , $g(\beta) \neq 0$. Take an x-admissible α and let $y = xu_{-\alpha}$. Then 0 is y-admissible since $f(\alpha, -\alpha) + g(\alpha) - g(-\alpha) = 2g(\alpha) \neq 0$, so $y \neq 0$ and $\lambda(y) = \lambda(x)$. Thus $\beta - \alpha$ is y-admissible for every x-admissible β , and since 0 is y-admissible, $g(\beta - \alpha) = 0$ for every x-admissible β . Now if α and β are x-admissible and $\alpha \neq \beta$, then since $\lambda(xu_{\beta}) < \lambda(x)$, we have $xu_{\beta} = 0, 0 = f(\alpha, \beta) + g(\alpha - \beta) = g(\alpha) [h(\beta - \alpha)]$, so $g(\alpha - \beta) = h(\alpha - \beta) = 0$ and $\alpha = \beta$, a contradiction.

Hence $\lambda(x) = 1$ and M contains some u_{α} . If $g(\alpha) = 0$ then for β such

that $g(\beta) \neq 0$, M contains $u_{\alpha}u_{\beta} = -g(\beta) [h(\alpha)+1]u_{\alpha+\beta}\neq 0$, so M contains a u_{γ} with $g(\gamma) \neq 0$. Now if β is such that $g(\beta) = 0$ then M contains u_{β} , since $u_{\gamma}u_{-\gamma+\beta} = g(\gamma) [h(\beta)+2]u_{\beta}\neq 0$. In particular M contains u_0 and so also any u_{α} with $g(\alpha) \neq 0$, since $u_0u_{\alpha} = -g(\alpha)u_{\alpha}\neq 0$. Therefore M = L, and we have proved the following theorem.

THEOREM 1. The algebra L(G, g, f) defined by (1) is a simple Lie algebra if and only if the characteristic is not two, g does not vanish identically and there is an additive mapping h from G to F for which (2) holds and for which there is no nonzero δ in G with $g(\delta) = 0$ and $h(\delta) = 0$ or 2.

From the conditions of this theorem on the mappings g and h, the following result may be proved easily.

COROLLARY 1. Suppose that F is a field of characteristic 0, of degree d (finite or infinite) over the rationals. Then if L(G, g, f) is a simple Lie algebra over F, the group G must be torsion-free. Let G be a given torsionfree abelian group and let r be the rank (the maximum number of linearly independent elements) of G. Then there exists a simple Lie algebra L(G, g, f) over F if and only if $2d \ge r$, when G is not divisible, or 2d $\ge r+1$, when G is divisible.

Similar statements hold when F has prime characteristic p. In that case G must be an elementary p-group in order for L(G, g, f) to be a simple Lie algebra. The Lie algebras L(G, g, f) were shown to be simple when g is an isomorphism by Albert and Frank in [1, p. 138]. In the finite dimensional case the simple Lie algebras L(G, g, f) may be shown to be the same up to isomorphism as the p^n -dimensional simple Lie algebras considered by Jennings and Ree in [3].

3. The algebra of derivations of L(G, g, f). We shall henceforth assume that any algebra L(G, g, f) considered is a simple Lie algebra, and in particular that a mapping h satisfying the conditions of Theorem 1 is given.

Now suppose that D is a derivation of L(G, g, f) and let $c(\alpha, \gamma)$ be the coefficient of $u_{\alpha+\gamma}$ in $u_{\alpha}D$, that is,

$$D: u_{\alpha} \to \sum_{\gamma \in G} c(\alpha, \gamma) u_{\alpha+\gamma} = \sum_{\gamma \in G} c(\alpha, -\alpha + \gamma) u_{\gamma},$$

the sums being finite of course. Since D is a derivation, $(u_{\gamma}u_{\epsilon})D = (u_{\gamma}D)u_{\epsilon} + u_{\gamma}(u_{\epsilon}D)$, that is, with ϕ (α, β) denoting $f(\alpha, \beta) + g(\alpha - \beta)$, $\sum_{\zeta \in \mathcal{G}} \left\{ \phi(\gamma, \epsilon)c(\gamma + \epsilon, -\gamma - \epsilon + \zeta) - \phi(\zeta - \epsilon, \epsilon)c(\gamma, -\gamma - \epsilon + \zeta) - \phi(\gamma, -\gamma + \zeta)c(\epsilon, -\gamma - \epsilon + \zeta) \right\} u_{\zeta} = 0$ for all γ and ϵ in G. Taking $\zeta = \gamma + \epsilon + \theta$ this gives

(3)
$$[f(\gamma, \epsilon) + g(\gamma - \epsilon)]c(\gamma + \epsilon, \theta) = [f(\gamma + \theta, \epsilon) + g(\gamma - \epsilon + \theta)]c(\gamma, \theta) + [f(\gamma, \epsilon + \theta) + g(\gamma - \epsilon - \theta)]c(\epsilon, \theta)$$

for all γ , ϵ and θ in G.

LEMMA 1. If $\theta \neq 0$ then

(4)
$$[f(\beta, \theta) + g(\beta - \theta)]c(\alpha, \theta) = [f(\alpha, \theta) + g(\alpha - \theta)]c(\beta, \theta)$$

for any α and β in G.

In this proof we denote $c(\gamma, \theta)$ by c_{γ} for any γ . The proof is divided into two cases.

CASE I. $g(\theta) \neq 0$. Taking $\gamma = \alpha$ and $\epsilon = 0$ in (3), we get $0 = g(\theta)c_{\alpha} + [f(\alpha, \theta) + g(\alpha - \theta)]c_0$, i.e.,

$$c_{\alpha} = [f(\alpha, \theta) + g(\alpha - \theta)][-g(\theta)]^{-1}c_0$$

which, together with the similar result for β , gives (4).

CASE II. $g(\theta) = 0, \theta \neq 0$. If $g(\alpha) = g(\beta) = 0$ then both sides of (4) vanish, so we may assume that, say, $g(\alpha) \neq 0$. With $\gamma = \alpha$ and $\epsilon = 0$, (3) gives $g(\alpha) [h(\theta) + 1] c_0 = 0$, i.e., $c_0 = 0$. Now let ζ and η be any nonzero elements of *G* such that $g(\zeta) \neq 0$ and $g(\eta) = 0$. Then (3) with $\gamma = \zeta$ and $\epsilon = -\zeta$ gives $0 = g(\zeta) [h(\theta) + 2] \{c_{\zeta} + c_{-\zeta}\}$, i.e.,

$$c_{-\varsigma} = -c_{\varsigma}.$$

Also by (3), $g(\zeta) [h(\eta)+1] c_{\zeta+\eta} = g(\zeta) [h(\eta)+1] c_{\zeta} + g(\zeta) [h(\eta+\theta)+1] c_{\eta}$, i.e.,

 $c_{\zeta+\eta} = c_{\zeta} + [h(\eta) + 1]^{-1} [h(\eta + \theta) + 1] c_{\eta},$

while with $\gamma = -\zeta$ and $\epsilon = \zeta + \eta$, (3) gives

$$-g(\zeta)[h(\eta) + 2]c_{\eta} = -g(\zeta)[h(\eta + \theta) + 2]\{c_{-\zeta} + c_{\zeta+\eta}\}$$

i.e.,

$$c_{\zeta+n} = -c_{-\zeta} + [h(\eta+\theta)+2]^{-1}[h(\eta)+2]c_{\eta}.$$

Hence $[h(\eta+\theta)+2][h(\eta+\theta)+1]c_{\eta} = [h(\eta)+1][h(\eta)+2]c_{\eta}$, so that $h(\theta)[h(2\eta+\theta)+3]c_{\eta}=0$. Thus $c_{\eta}=0$ and $c_{\zeta+\eta}=c_{\zeta}$ if $h(2\eta+\theta)\neq-3$ while if $h(2\eta+\theta)=-3$ then, since $h(4\eta+\theta)\neq-3$ and $h(-2\eta+\theta)\neq-3$, we have $c_{\zeta+\eta}=c_{\zeta-\eta+2\eta}=c_{\zeta-\eta}=c_{\zeta}$, and again $c_{\eta}=0$. It follows that

$$c_{\alpha+\beta+\theta} = c_{\alpha+\beta}, \qquad c_{\beta+\theta} = c_{\beta}.$$

Now with $\gamma = \alpha$ and $\epsilon = \beta + \theta$ in (3) we have

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$$[f(\alpha, \beta + \theta) + g(\alpha - \beta)]c_{\alpha+\beta}$$

= $[f(\alpha + \theta, \beta + \theta) + g(\alpha - \beta)]c_{\alpha} + [f(\alpha, \beta + 2\theta) + g(\alpha - \beta)]c_{\beta}.$

Subtracting from this (3) with $\gamma = \alpha$ and $\epsilon = \beta$ we get $f(\alpha, \theta)c_{\alpha+\beta} = f(\alpha, \theta) \{c_{\alpha}+c_{\beta}\}$, so that $c_{\alpha+\beta}=c_{\alpha}+c_{\beta}$. Now (3) with $\gamma = \alpha$ and $\epsilon = \beta$ gives

$$f(\beta, \theta)c_{\alpha} = f(\alpha, \theta)c_{\beta}.$$

Multiplying both sides of this by $[h(\theta)]^{-1}[h(\theta)+1]$, we get (4) for this case also, which proves the lemma.

LEMMA 2. The derivation D differs by an inner derivation D' from a derivation for which the coefficients $c(\alpha, \theta)$ vanish for all nonzero θ .

Indeed for any nonzero θ we may take an α such that $f(\alpha, \theta) + g(\alpha - \theta) \neq 0$ and set $k_{\theta} = [f(\alpha, \theta) + g(\alpha - \theta)]^{-1}c(\alpha, \theta)$. By Lemma 1, k_{θ} is well defined. Thus if $g(\theta) \neq 0$ then $k_{\theta} = -[g(\theta)]^{-1}c(0, \theta)$, and since $c(0, \theta) \neq 0$ for only finitely many θ , there are only finitely many θ such that $g(\theta) \neq 0$ and $k_{\theta} \neq 0$. Similarly, taking α such that $g(\alpha) \neq 0$, we find that there are also only finitely many θ such that $g(\theta) = 0$ and $k_{\theta} \neq 0$. Therefore we may consider the right multiplication by $\sum_{\theta \neq 0} k_{\theta} u_{\theta}$. Taking this to be D', and noting that by Lemma 1 if $f(\alpha, \theta) + g(\alpha - \theta) = 0$ then $c(\alpha, \theta) = 0$ (for nonzero θ), we see that the lemma holds.

Now let d be any additive function of G to F. Then it is easy to see that the linear transformation D_d determined by the mapping of the basis elements

 $u_{\alpha} \rightarrow d(\alpha) u_{\alpha}$

is a derivation of L(G, g, f).

THEOREM 2. The algebra of derivations of the simple Lie algebra L(G, g, f) is spanned by the inner derivations together with all the derivations D_d .

What remains to be proved is that

(5)
$$c_{\alpha+\beta} = c_{\alpha} + c_{\beta}$$

for any α and β , where, for any γ , c_{γ} denotes $c(\gamma, 0)$. If $f(\alpha, \beta) + g(\alpha - \beta) \neq 0$, (5) follows directly from (3) with $\theta = 0$. But if $f(\alpha, \beta) + g(\alpha - \beta) = 0$ then we may pick a γ such that the expressions

$$f(\alpha + \beta, \gamma) + g(\alpha + \beta - \gamma), \qquad f(\alpha, \beta + \gamma) + g(\alpha - \beta - \gamma),$$

$$f(\beta, \gamma) + g(\beta - \gamma)$$

are all nonzero, for if $g(\alpha) = 0$ (so that $g(\beta) = 0$ also) then any γ for which $g(\gamma) \neq 0$ will do, while if $g(\alpha) \neq 0$ we may take $\gamma = 2\beta$ unless $\alpha = \beta$. When $g(\alpha) \neq 0$ and $\alpha = \beta$ (and the characteristic is not 3) we may take $\gamma = -\alpha$. Now with such a γ we have

$$c_{\alpha+\beta} + c_{\gamma} = c_{\alpha+\beta+\gamma} = c_{\alpha} + c_{\beta+\gamma} = c_{\alpha} + c_{\beta} + c_{\gamma}$$

(when the characteristic is 3 and $\alpha = \beta$ one argues that $c_{2\alpha} = c_{-\alpha} = -c_{\alpha} = 2c_{\alpha}$). Thus (5) holds and the theorem is proved.

Noting that the derivation sending u_{α} to $g(\alpha)u_{\alpha}$ is inner, we obtain the following result.

COROLLARY 2. If G has finite rank n then the algebra of outer derivations of L(G, g, f) is an abelian Lie algebra of dimension n-1.

4. Criteria for isomorphism. Henceforth we shall restrict our consideration to simple Lie algebras L(G, g, f) of characteristic zero, so that G must be torsion-free.

A derivation D of an algebra L is locally algebraic if and only if it is true that for every x in L the set $\{xD^i: i=1, 2, \cdots\}$ lies in a finite-dimensional subspace (depending on x) of L.

LEMMA 3. The only locally algebraic derivations of L(G, g, f) are the derivations D_d .

The derivations D_d are obviously locally algebraic. Now suppose that D is a locally algebraic derivation. By Theorem 2, $D = R_y + D_d$, where R_y is the right multiplication by $y = \sum_{\gamma} a_{\gamma} u_{\gamma}$, for some y and d. Suppose that some nonzero γ is y-admissible. We may simply order G in such a way that this $\gamma > 0$. Call u_{ϵ} the leading term in an element z of L(G, g, f) if ϵ is the greatest z-admissible element of G, and let u_{α} be the leading term in y. Thus $\alpha > 0$. We shall find a β such that the leading term in $(u_{\beta})D^i$ is $u_{\beta+i\alpha}$, contradicting the assumption that D is locally algebraic. Indeed if $g(\alpha) = 0$ we may take β to be any positive element of G with $g(\beta) \neq 0$, since then the coefficient of $u_{\beta+i\alpha}$ in $(u_{\beta})D^i$ is $[g(\beta)]^i[h(\alpha)+1]^i \neq 0$, while if $g(\alpha) \neq 0$ we may take β to be 2α . Thus y must be a scalar multiple of u_0 , and the lemma is proved.

Since for any distinct elements α and β there is an additive mapping d of G to F such that $d(\alpha) \neq d(\beta)$, we have the following result.

LEMMA 4. The only elements of L(G, g, f) which are characteristic vectors for all locally algebraic derivations are the scalar multiples of all the elements u_{α} .

Now suppose that σ is an isomorphism of one algebra L(G, g, f) onto another, L(G', g', f'). We shall determine the relations between

G, g and f on the one hand, and G', g' and f' on the other. It follows from Lemma 4 that for every α in G there is an element α^{σ} in G' such that

(6)
$$(u_{\alpha})\sigma = cl_{\alpha}u_{\alpha}\sigma$$

where l_{α} is a nonzero scalar (depending on α and σ) and c is a fixed scalar chosen so that $l_0 = 1$. The induced mapping $\sigma: \alpha \rightarrow \alpha^{\sigma}$ of G into G' is one-to-one and onto. Since $(u_{\gamma}u_{\epsilon})\sigma = [(u_{\gamma})\sigma][(u_{\epsilon})\sigma]$,

(7)
$$cl_{\gamma+\epsilon} [f(\gamma, \epsilon) + g(\gamma - \epsilon)] u_{(\gamma+\epsilon)^{\sigma}} = c^2 l_{\gamma} l_{\epsilon} [f'(\gamma^{\sigma}, \epsilon^{\sigma}) + g'(\gamma^{\sigma} - \epsilon^{\sigma})] u_{\gamma^{\sigma}+\epsilon^{\sigma}}$$

for all γ and ϵ in G. Hence if $f(\gamma, \epsilon) + g(\gamma - \epsilon) \neq 0$ then $(\gamma + \epsilon)^{\sigma} = \gamma^{\sigma} + \epsilon^{\sigma}$, so that, exactly as in the final part of the proof of Theorem 2, σ is always additive and therefore is an isomorphism of G onto G'.

Taking $\gamma = \alpha$ and $\epsilon = 0$ in (7) we get

(8)
$$g'(\alpha^{\sigma}) = c^{-1}g(\alpha)$$

for any α in G. Now taking $\epsilon = -\gamma$ in (7), we have $l_{-\gamma} = l_{\gamma}^{-1}$ for any γ , and taking $\gamma = 2\zeta$ and $\epsilon = -\zeta$ we have $l_{\zeta} = l_{2\zeta}l_{-\zeta}$, i.e., $l_{2\zeta} = l_{\zeta}^2$ for any ζ in G. If $f(\alpha, \beta) + g(\alpha - \beta) = 0$ then, by (7) and (8), $cf'(\alpha^{\sigma}, \beta^{\sigma}) + g(\alpha - \beta) = 0$ and

(9)
$$f'(\alpha^{\sigma},\beta^{\sigma}) = c^{-1}f(\alpha,\beta).$$

Similarly, if $f(\alpha, \beta) - g(\alpha - \beta) = 0$ or $4f(\alpha, \beta) + 2g(\alpha - \beta) = 0$ then by taking $\gamma = -\alpha$ and $\epsilon = -\beta$, or $\gamma = 2\alpha$ and $\epsilon = 2\beta$, in (7), we have (9) again. Now suppose that the expressions $f(\alpha, \beta) + g(\alpha - \beta)$, $f(\alpha, \beta) - g(\alpha - \beta)$ and $4f(\alpha, \beta) + 2g(\alpha - \beta)$ are nonzero. By (7) and (8) we have

$$l_{-\alpha-\beta} = l_{-\alpha}l_{-\beta}[cf'(\alpha^{\sigma},\beta^{\sigma}) - g(\alpha-\beta)][f(\alpha,\beta) - g(\alpha-\beta)]^{-1}$$

while on the other hand

$$l_{-\alpha-\beta} = (l_{\alpha+\beta})^{-1} = l_{\alpha}^{-1}l_{\beta}^{-1}[cf'(\alpha^{\sigma},\beta^{\sigma}) + g(\alpha-\beta)]^{-1}[f(\alpha,\beta) + g(\alpha-\beta)].$$

Hence $[cf'(\alpha^{\sigma}, \beta^{\sigma})]^2 = [f(\alpha, \beta)]^2$. Now similarly by expanding $l_{2\alpha+2\beta}$ in two different ways one may see that (9) always holds. It then follows that $l_{\alpha+\beta} = l_{\alpha}l_{\beta}$ for any α and β in G. We have thus proved one direction of the following theorem. The converse is clear from (7).

THEOREM 3. A linear mapping $\sigma: L(G, g, f) \rightarrow L(G', g', f')$ of one of the simple Lie algebras L(G, g, f) of characteristic zero onto another is an isomorphism if and only if there is an induced isomorphism $\sigma: \alpha \rightarrow \alpha^{\sigma}$ of G onto G', a nonzero scalar c and a homomorphism $l: \alpha \rightarrow l_{\alpha}$ of G into the multiplicative group F^* of the base field, such that (6), (8) and (9) hold for all α and β in G.

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This result in particular applies to automorphisms of L(G, g, f). Thus if G has rank one then the automorphism group of L(G, g, f)is a semidirect product of a normal subgroup A and a subgroup B, where A is isomorphic to the group of homomorphisms of G into F^* and B is isomorphic to the group of automorphisms of G. Also if G is a free abelian group on n generators $\alpha_1, \dots, \alpha_n$ and the elements $g(\alpha_i)g(\alpha_j)$ $(1 \le i \le j \le n)$ of F are linearly independent over the rationals, then the automorphism group of L(G, g, f) is again a semidirect product of groups A and B, where A is an n-fold direct product of F^* , and B has order 2 or 1 according to whether f vanishes identically or not.

BIBLIOGRAPHY

1. A. A. Albert, and M. S. Frank, Simple Lie algebras of characteristic p, Univ. e Politec. Torino. Rend. Sem. Mat. vol. 14 (1954–1955) pp. 117–139.

2. Graham Higman, The units of group-rings, Proc. London Math. Soc. vol. 46 (1939-1940) pp. 231-248.

3. S. A. Jennings and Rimhak Ree, On a family of Lie algebras of characteristic p, Trans. Amer. Math. Soc. vol. 84 (1957) pp. 192-207.

4. Rimhak Ree, On generalized Witt algebras, Trans. Amer. Math. Soc. vol. 83 (1956) pp. 510-546.

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