THE DISTRIBUTION OF a—POINTS OF AN ENTIRE FUNCTION

S. K. SINGH AND S. H. DWIVEDI

1. Let $f(z)$ be an entire function of order ρ $(0<\rho<\infty)$ and lower order λ $(0\leqq\lambda<\infty)$. It is known that corresponding to every entire function of finite nonzero order there exists a function $\rho(r)$ called its proximate order having the following properties:

(1.1) $\rho(r)$ is real, continuous and piecewise differentiable,

(1.2) $\rho(r)\to\rho$ as $r\to\infty$,

(1.3) $r\rho'(r)\log r\to 0$ as $r\to\infty$,

$$\begin{aligned}\log M(r,f) &\leqq r^{\rho(r)} \text{ for } r\geqq r_0\\ &= r^{\rho(r)} \text{ for a sequence of values of } r.\end{aligned}\tag{1.4}$$

2. S. M. Shah [2] has proved the existence of a function $\lambda(r)$ for an entire function of lower order λ $(0\leqq\lambda<\infty)$ analogous to $\rho(r)$, having the following properties:

(2.1) $\lambda(r)$ is a non-negative, continuous function of r for $r\geqq r_0$.

(2.2) $\lambda(r)$ is differentiable except at isolated points at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist.

(2.3) $\lambda(r)\to\lambda$ as $r\to\infty$.

(2.4) $r\lambda'(r)\log r\to 0$ as $r\to\infty$.

$$\begin{aligned}\log M(r,f) &\geqq r^{\lambda(r)} \text{ for } r\geqq r_0\\ &= r^{\lambda(r)} \text{ for a sequence of values of } r.\end{aligned}\tag{2.5}$$

3. In this note we prove a number of results applying the properties of $\lambda(r)$ and $\rho(r)$. In what follows we shall take $0<\lambda<\infty$. From properties (2.1)–(2.5) of $\lambda(r)$ we can easily deduce that $r^{\lambda(r)}$ is an increasing function of $r(r\geqq r_0)$, for

$$\frac{d}{dr}\left(r^{\lambda(r)}\right) = (o(1)+\lambda(r))r^{\lambda(r)-1} > 0 \text{ for } r\geqq r_0.$$

With the usual notations of $\log M(r,f)$, $n(r,a)$ and $N(r,a)$ we prove the following theorems:

THEOREM 1. *If*

$$\limsup_{r\to\infty}\frac{\log M(r,f)}{r^{\lambda(r)}}<\infty\tag{3.1}$$

and

Received by the editors December 6, 1957 and, in revised form, January 28, 1958.

$$(3.2) \qquad \frac{N(r, a)}{r^{\lambda(r)}} \to 0 \text{ a } \quad r \to \infty,$$

then for

$$x \neq a$$

$$\text{(i)} \qquad 0 < \liminf_{r\to\infty} N(r, x)/r^{\lambda(r)} \leqq 1,$$

$$\text{(ii)} \quad ((h-1)/(h+1))(1/h^{\lambda}) \leqq \limsup_{r\to\infty} N(r, x)/r^{\lambda(r)}$$

$$\leqq (1/\lambda) \limsup_{r\to\infty} n(r, x)/r^{\lambda(r)} < \infty$$

where $h = (1 + (1+\lambda^2)^{1/2})/\lambda$.

THEOREM 2.

$$\text{(i)} \qquad \liminf_{r\to\infty} n(r)/r^{\lambda(r)} \leqq \lambda,$$

$$\text{(ii)} \qquad \liminf_{r\to\infty} n(r)/r^{\rho(r)} \leqq \rho.$$

THEOREM 3.
(i) *If*

$$\lim_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \textit{ exists, then } \lim_{r\to\infty} \frac{n(r, x)}{r^{\lambda(r)}} = \lambda \lim_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}}.$$

(ii) *If*

$$\lim_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}} \textit{ exists, then } \lim_{r\to\infty} \frac{n(r, x)}{r^{\rho(r)}} = \rho \lim_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}}.$$

THEOREM 4. *If $f(z)$ be an entire function of finite nonzero order for which*

$$\frac{n(r, a)}{\log M(r, f)} \to 0 \qquad \textit{as } r \to \infty,$$

then

$$\liminf_{r\to\infty} \frac{N(r, a)}{\log M(r, f)} = 0.$$

We observe that the above theorem does not hold if $\rho = 0$. For instance consider

$$f(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{e^n}\right).$$

Then $f(z)$ is an entire function of zero order for which

$$n(r, 0) \sim \log r,$$
$$N(r, 0) \sim 1/2(\log r)^2,$$
$$\log M(r, f) \sim N(r, 0);$$

hence

$$\frac{n(r, 0)}{\log M(r,f)} \to 0, \quad \text{but} \quad \frac{N(r, 0)}{\log M(r,f)} \to 1.$$

As another example we can take any polynomial $P(z)$ then

$$\frac{n(r, 0)}{\log M(r, P)} \to 0, \quad \text{but} \quad \frac{N(r, 0)}{\log M(r, P)} \to K \qquad (K > 0).$$

THEOREM 5. *Let $f(z)$ be an entire function of order ρ $(0<\rho<\infty)$ such that*

(i) $$\liminf_{r\to\infty} \log M(r, f)/r^{\rho(r)} > 0,$$

(ii) $$\lim_{r\to\infty} N(r, a)/r^{\rho(r)} = 0,$$

then

$$0 < \liminf_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}} \leqq \limsup_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}} \leqq 1 \qquad \textit{for all } x \neq a.$$

In the above theorem Condition (1) namely

$$\liminf_{r\to\infty} \log M(r, f)/r^{\rho(r)} > 0$$

is essential, because there exist entire functions $f(z)$ for which $\liminf_{r\to\infty} N(r, a_\nu)/r^{\rho(r)} = 0$ for $\nu = 1, 2, \cdots, k$.

For instance see S. M. Shah and S. K. Singh [**4**, Theorem I(ii)]. There

$$\lambda_1(-a_\nu) = \liminf_{r\to\infty} \frac{\log n(r, f + a_\nu)}{\log r} < \lambda(f) = \liminf_{r\to\infty} \frac{\log \log M(r)}{\log r}$$
$$(\nu = 1, 2, \cdots, k)$$

and since

$$\liminf_{r\to\infty} \frac{\log n(r, f + a_\nu)}{\log r} = \liminf_{r\to\infty} \frac{\log N(r, f + a_\nu)}{\log r}$$

so $N(r, f+a_\nu) < r^{\lambda_1(-a_\nu)+\epsilon}$ for a sequence of values of r, also $\log M(r, f) > r^{\lambda-\epsilon}$ for all $r \geqq r_0$, so $\liminf_{r\to\infty} N(r, f+a_\nu)/\log M(r, f) = 0$; and hence a fortiori $\liminf_{r\to\infty} N(r, f+a_\nu)/r^{\rho(r)}=0$ $(\nu=1, 2, \cdots, k)$.

4. LEMMA 1. $(hr)^{\lambda(hr)} \sim h^\lambda r^{\lambda(r)}$.

LEMMA 2. $\int_{r_0}^r t^{\lambda(t)-1}dt \sim r^{\lambda(r)}/\lambda$.

PROOF OF LEMMA 1. It is sufficient to prove that $r^{\lambda(hr)} \sim r^{\lambda(r)}$. Now

$$\lambda(hr) - \lambda(r) = \int_r^{hr} \lambda'(t)dt = o\left(\int_r^{hr} \frac{dt}{t \log t}\right) = o\left(\frac{1}{\log r}\right).$$

Hence

$$r^{\lambda(hr)-\lambda(r)} \to 1.$$

Proof of Lemma 2 is similar to [1, Lemma 4, p. 58].

PROOF OF THEOREM 1(i). From (2.5) we have

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda(r)}} = 1. \tag{4.1}$$

Hence the right hand inequality is obvious as $N(r, x) \leqq \log M(r, f)$. Also clearly $N(r, x) > N(r, a)$, $(x \neq a)$ for if $N(r, x) \leqq N(r, a)$, then from Nevanlinna's second theorem

$$\begin{aligned} T(r,f) &< N(r, a) + N(r, x) + O(\log r) \\ &\leqq 2N(r, a) + O(\log r), \\ \frac{T(r,f)}{r^{\lambda(r)}} &\leqq \frac{2N(r, a)}{r^{\lambda(r)}} + o(1). \end{aligned}$$

Hence, $T(r, f)/r^{\lambda(r)} \to 0$ and as $T(r, f) \leqq \log M(r, f) \leqq 3T(2r, f)$; so $\log M(r, f)/r^{\lambda(r)} \to 0$ as $r \to \infty$; this contradicts (4.1).

Hence, appealing to Nevanlinna's second theorem again we have

$$\begin{aligned} T(r,f) &< 2N(r, x) + O(\log r), \\ \frac{T(r,f)}{r^{\lambda(r)}} &< \frac{2N(r, x)}{r^{\lambda(r)}} + o(1). \end{aligned}$$

Hence, $2N(2r, x)/(2r)^{\lambda(2r)} > T(2r, f)/(2r)^{\lambda(2r)} > A \log M(r, f)/r^{\lambda(r)}$ and

$$\liminf_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \geqq A \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda(r)}} = A > 0.$$

(ii) Now, $\limsup_{r\to\infty} N(r, a)/r^{\lambda(r)} + \limsup_{r\to\infty} N(r, x)/r^{\lambda(r)} \geqq \limsup_{r\to\infty} T(r,f)/r^{\lambda(r)}$ and as $\limsup_{r\to\infty} N(r, a)/r^{\lambda(r)} = 0$, so

$$\limsup_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\lambda(r)}} \cdot$$

Also, $T(hr, f) > (h-1)/(h+1) \log M(r, f)$, $(h>1)$ so,

$$\limsup_{r\to\infty} \frac{T(hr,f)}{(hr)^{\lambda(hr)}} \geqq \frac{h-1}{h+1}\,\frac{1}{h^{\lambda}} \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda(r)}} \geqq \frac{h-1}{h+1}\,\frac{1}{h^{\lambda}},$$

since, $\limsup_{r\to\infty} \log M(r, f)/r^{\lambda(r)} \geqq 1$ by (4.1).

Now choosing the best possible value of h which is

$$h = (1 + (1 + \lambda^2)^{1/2})/\lambda$$

we have

$$\frac{h-1}{h+1}\,\frac{1}{h^{\lambda}} \leqq \limsup_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \cdot$$

Let now $\limsup_{r\to\infty} n(r, x)/r^{\lambda(r)} = H$, then,

$$N(r, x) < \int_{r_0}^{r} (H+\epsilon) t^{\lambda(t)-1} dt \sim \frac{H+\epsilon}{\lambda} r^{\lambda(r)}.$$

Hence,

$$\limsup_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \leqq \frac{H}{\lambda} = \frac{1}{\lambda} \limsup_{r\to\infty} \frac{n(r, x)}{r^{\lambda(r)}}.$$

Further from Jensen's theorem we have

$$n(r, x) \log 2 \leqq \int_{r}^{2r} \frac{n(t, x)}{t}\, dt < \int_{0}^{2r} \frac{n(t, x)}{t}\, dt < \log M(2r, f).$$

Hence, $n(r, x) \log 2/r^{\lambda(r)} < [(\log M(2r, f)/(2r)^{\lambda(2r)})]((2r)^{\lambda(2r)}/r^{\lambda(r)})$; so,

$$\limsup_{r\to\infty} \frac{n(r, x)}{r^{\lambda(r)}} \leqq A \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda(r)}} < \infty.$$

PROOF OF THEOREM 2(i). Let $\liminf_{r\to\infty} n(r)/r^{\lambda(r)} = H$, then

$$n(r) > (H-\epsilon) r^{\lambda(r)} \qquad \text{for } r \geqq r_0;$$

so,

$$N(r) > \int_{r_0}^{r} (H-\epsilon) t^{\lambda(t)-1} dt$$

$$\sim \frac{(H-\epsilon) r^{\lambda(r)}}{\lambda} = \frac{(H-\epsilon)}{\lambda} \log M(r, f)$$

for a sequence of values of r.

Hence, $\limsup_{r\to\infty} N(r)/\log M(r,f) \geqq H/\lambda$ and so

$$H/\lambda \leqq \limsup_{r\to\infty} N(r)/\log M(r,f) \leqq 1.$$

Hence, $H \leqq \lambda$.

The proof of (ii) is similar.

PROOF OF THEOREM 3(i). Let $\lim_{r\to\infty} N(r,x)/r^{\lambda(r)} = M$. Set $N(r,x) = N(r)$, then

$$(M-\epsilon)r^{\lambda(r)} < N(r) < (M+\epsilon)r^{\lambda(r)}.$$

$$\begin{aligned}\int_r^{r(1+\alpha)} \frac{n(t)}{t}\,dt &= N(r+r\alpha) - N(r) < (M+\epsilon)(r+r\alpha)^{\lambda(r+r\alpha)} \\ &\qquad - (M-\epsilon)r^{\lambda(r)} \\ &\sim (M+\epsilon)(1+\alpha)^{\lambda} r^{\lambda(r)} - (M-\epsilon)r^{\lambda(r)} \\ &= r^{\lambda(r)}\{(M+\epsilon)(1+\alpha)^{\lambda} - (M-\epsilon)\} \\ &= r^{\lambda(r)}\left\{(M+\epsilon)\left(1 + \lambda\alpha + \frac{\lambda(\lambda-1)}{2!}\alpha^2 + \cdots\right) - M + \epsilon\right\} \\ &= r^{\lambda(r)}\left\{\left(M\alpha\lambda + \frac{M\lambda(\lambda-1)}{2!}\alpha^2 + \cdots\right) + 2\epsilon + \epsilon\lambda\alpha\right\}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{n(r)}{r^{\lambda(r)}}\,\frac{\alpha}{1+\alpha} &< \int_r^{r(1+\alpha)} \frac{n(t)}{t}\,dt \\ &< \left\{M\lambda\alpha + 2\epsilon + \epsilon\lambda\alpha + \frac{M\lambda(\lambda-1)\alpha^2}{2!} + \cdots\right\},\end{aligned}$$

$$\frac{n(r)}{r^{\lambda(r)}} < (1+\alpha)\left\{M\lambda + \frac{2\epsilon}{\alpha} + \epsilon\lambda + \frac{M\lambda(\lambda-1)}{2!}\alpha + \cdots\right\}.$$

Setting first $\alpha = \epsilon^{1/2}$ and then making $\epsilon \to 0$, we get $\limsup_{r\to\infty} n(r)/r^{\lambda(r)} \leqq M\lambda$. Similarly we can prove that $\liminf_{r\to\infty} n(r)/r^{\lambda(r)} \geqq M\lambda$, and the first part of the theorem follows. The proof of the second part is similar.

We omit the proofs of Theorems 4 and 5.

5. We know that for all values of a

$$\limsup_{r\to\infty} \frac{n(r,a)}{r^{\rho(r)}} < \infty. \tag{5.1}$$

The question naturally arises whether (5.1) is still true if we replace $\rho(r)$ by $\lambda(r)$. We show that this is not so. Below we give an example in which $\limsup_{r\to\infty} n(r, a)/r^{\lambda(r)} = \infty$. Take, $f(z) = \prod_1^\infty (1+(z/\Delta_n)^{k\mu_n})$ where $k = [\rho]+1$, $\mu_n = \Delta_n^{\rho+\epsilon}$, $\Delta_n = n^{nn}$. Then,

$$\limsup_{r\to\infty} n(r, 0)/\log M(r, f) = \infty$$

(see S. M. Shah [3]). Now, since $\log M(r, f) \geqq r^{\lambda(r)}$ for $r \geqq r_0$, so $\limsup_{r\to\infty} n(r, 0)/r^{\lambda(r)} = \infty$.

References

1. M. L. Cartwright, *Integral functions*, Cambridge, 1956.

2. S. M. Shah, *A note on lower proximate orders*, J. Indian Math. Soc. vol. 12 (1948) pp. 31–32.

3. ———, *A note on maximum modulus and the zeros of an integral function*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 909–912.

4. S. M. Shah and S. K. Singh, *Borel's theorem on a—points and exceptional values of entire and meromorphic functions*, Math. Z. vol. 59 (1953) pp. 88–93.

D. S. College, Aligarh, India and
D. A. V. College, Aligarh, India