

A REMARK CONCERNING A THEOREM OF B. FRIEDMAN

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In [1], the following theorem is stated: let T be a densely defined linear operator with closed range in the Hilbert space \mathcal{H} , with a densely defined adjoint T^* also having closed range. Let ϕ, ψ be vectors in \mathcal{H} , and let $\phi \otimes \psi$ be the operator defined by $\phi \otimes \psi(x) = (x, \phi)\psi$. Then $T + \phi \otimes \psi$ also has closed range.

Of course, the fact that T^* is densely defined implies that T is pre-closed; but an examination of the proof shows that it actually requires that T be a *closed* operator. Under this assumption, a simpler proof can be given; and the need for some such condition will be shown by example.

THEOREM. *Let T be a closed, densely defined operator with closed range. Then $S = T + \phi \otimes \psi$ also has a closed range.*

PROOF. The nullspace \mathfrak{N}_T of a closed operator T is closed, and its domain \mathfrak{D}_T is the sum of the two subspaces \mathfrak{N}_T and $\mathfrak{D}_T \cap \mathfrak{N}_T^\perp = \mathfrak{S}_T$, since x in \mathfrak{D}_T can be written $(x - P_{\mathfrak{N}_T}x) + P_{\mathfrak{N}_T}x$ (where $P_{\mathfrak{N}_T}$ denotes the orthogonal projection on the subspace \mathfrak{N}_T). If we use the graph norm on \mathfrak{D}_T , given by the inner product $\langle x, y \rangle = (x, y) + (Tx, Ty)$, then \mathfrak{N}_T and \mathfrak{S}_T are complete, and \mathfrak{D}_T is their Hilbert space direct sum.

T restricted to \mathfrak{S}_T is a 1-1 continuous operator from \mathfrak{S}_T (in the graph norm) to the range \mathfrak{R}_T of T . The closed graph theorem tells us that its inverse R is continuous, as an operator from the Hilbert space \mathfrak{R}_T to \mathfrak{S}_T . Now, the orthogonal complement $[\phi]^\perp$ of ϕ in \mathcal{H} is closed in \mathcal{H} . Thus its intersection with \mathfrak{S}_T is closed in the graph norm. Then $R^{-1}([\phi]^\perp \cap \mathfrak{S}_T) = T([\phi]^\perp \cap \mathfrak{S}_T)$ is closed in \mathcal{H} . $T([\phi]^\perp \cap \mathfrak{S}_T) = S([\phi]^\perp \cap \mathfrak{S}_T) \subset \mathfrak{R}_S \subset \mathfrak{R}_T + [\psi] = T(\mathfrak{S}_T) + [\psi]$. Now, the codimension of $T([\phi]^\perp \cap \mathfrak{S}_T)$ in $T(\mathfrak{S}_T) + [\psi]$ is at most two, so that of $T([\phi]^\perp \cap \mathfrak{S}_T)$ in \mathfrak{R}_S is again at most two. Since $T([\phi]^\perp \cap \mathfrak{S}_T)$ is closed, \mathfrak{R}_S is also closed.

REMARK 1. If T had been merely preclosed, but with closed range, then it is easy to see $\mathfrak{R}_{\bar{T}} = \mathfrak{R}_T$, so that $\bar{S} = \bar{T} + \phi \otimes \psi$ has closed range.

REMARK 2. Here is an example of an operator T which is densely defined, bounded, and has closed range, and whose adjoint T^* is therefore bounded and has closed range, but for which $S = T + \phi \otimes \psi$ will *not* have closed range, for certain ϕ and ψ .

Let \mathcal{H}_0 be a proper infinite-dimensional subspace of \mathcal{H} , and

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$\psi, \psi_1, \psi_2, \psi_3, \dots$ an orthonormal basis for \mathcal{H}_0 . Let $\psi'_n = n^{-1/2}((n-1)^{1/2}\psi + \psi_n)$. Then the set $\Psi = \{\psi, \psi'_1, \psi'_2, \dots\}$ is linearly independent, and $\|\psi'_n - \psi\|^2 \rightarrow 0$. Enlarge Ψ to a maximal linearly independent set Φ in \mathcal{H}_0 . Thus, the linear combinations of elements of Φ span \mathcal{H}_0 . Let \mathcal{K}_0 be the set of all linear combinations of elements of $\Phi - \{\psi\}$. Then \mathcal{K}_0 has the following properties:

(1) \mathcal{K}_0 is dense in \mathcal{H}_0 .

(2) $\psi \notin \mathcal{K}_0$.

(3) $\mathcal{K}_0 + [\psi] = \mathcal{H}_0$.

Let ϕ be any unit vector in \mathcal{H}_0^\perp . Let T be the restriction of $P_{\mathcal{H}_0} - \phi \otimes \psi$ to $\mathcal{K}_0 + \mathcal{H}_0^\perp$. Then clearly $\mathcal{R}_T \subset \mathcal{H}_0$. Further, $T|_{\mathcal{K}_0} = P_{\mathcal{H}_0}|_{\mathcal{K}_0}$, so $\mathcal{K}_0 \subset \mathcal{R}_T$. Finally, $T\phi = -\psi$, so $\mathcal{R}_T = \mathcal{H}_0$. Notice also that $T^* = P_{\mathcal{H}_0} - \psi \otimes \phi$, and so if $x = x_1 + x_2 + \alpha\psi$, where $x_1 \in \mathcal{H}_0 \cap [\psi]^\perp$, $x_2 \in \mathcal{H}_0^\perp$, then $T^*x = x_1 + \alpha(\psi - \phi)$. Thus $\mathcal{R}_T^* = (\mathcal{H}_0 \cap [\psi]^\perp) + [\psi - \phi]$, clearly closed. However, $T + \phi \otimes \psi = P_{\mathcal{H}_0}|_{\mathcal{K}_0}$ has \mathcal{K}_0 as its range.

REFERENCE

1. B. Friedman, *Operations with a closed range*, Comm. Pure Appl. Math. VIII, vol. 4 (1955) p. 539.

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