ON THE HILBERT MATRIX, II[1]

MARVIN ROSENBLUM

1. The Hilbert Matrix is $H_k = ((n+m+1-k)^{-1}), m, n = 0, 1, 2, \cdots,$ where k is a real number that is not a positive integer. It is known [**2**; **3**; **7**] that if $x_0, x_1, x_2, \cdots,$ is a sequence of complex numbers,[2] then

$$0 \leqq \sum_{n,m=0}^{\infty} (n + m + 1 - k)^{-1} x_n x_m^* \leqq M_k \sum_{n=0}^{\infty} | x_n |^2$$

where the best possible constant M_k is π for $k \leqq 1/2$ and $\pi | \csc \pi k |$ for $1/2 < k$. Thus, when considered as a linear operator on the complex sequential Hilbert space l_2, H_k is a bounded symmetric operator. Magnus [**8**] showed that the l_2 spectrum of H_0 is purely continuous and consists of the interval $[0, \pi]$. In this note we shall exhibit for each real k a monotone function $\rho_k(\lambda)$ and an isometric map V_k of l_2 onto $L^2(d\rho_k)$ such that $V_k H_k V_k^{-1}$ is a multiplication operator. This will allow us to determine the spectral nature of H_k.

In [**9**] we studied an isomorphism of l^2 with $L^2(0, \infty)$ that transforms the Hilbert operator H_k into an integral operator which we shall now denote by $\mathcal{H}_{k,1/2}$. It can be easily checked that $\mathcal{H}_{k,1/2}$ formally commutes with the differential operator L_k which is defined below. Indeed, we shall prove that $\mathcal{H}_{k,1/2} = \pi \operatorname{sech} \pi L_k^{1/2}$. Since L_k can be diagonalized by a now standard procedure so $\mathcal{H}_{k,1/2}$ and hence H_k can be diagonalized.

2. We first shall apply the Titchmarsh-Kodaira theory of singular differential operators [**11**; **4**], to the operator L_k, where $(L_k y)(x) = -(x^2 y'(x))' - (kx - x^2/4 + 1/4) y(x)$, $x \geqq 0$, k real. Suppose that λ is a complex number with positive imaginary part and $u = i\lambda^{1/2}$, $\pi < \arg u < 3\pi/2$. $L_k y = \lambda y$ has the linearly independent solutions $\alpha_k(x, \lambda) = W_{k,u}(x) x^{-1}$ and $\beta_k(x, \lambda) = [\Gamma(1 - 2u)]^{-1} \Gamma(1/2 - k - u) \cdot M_{k,-u}(x) x^{-1}$, where W and M are Whittaker functions [**1**, Chapter 6]. Considered as functions of x, $\alpha_k \in L^2(1, \infty)$, $\alpha_k \notin L^2(0, 1)$, $\beta_k \in L^2(0, 1)$, $\beta_k \notin L^2(1, \infty)$, and the Wronskian of α_k and β_k is 1. Thus L_k is of the limit point type at 0 and ∞ and has the Green's

---

Presented to the Society April 26, 1958; received by the editors July 25, 1957 and, in revised form, February 3, 1958.

[1] This research was partially supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF 49(638)-72 at the University of Virginia.

[2] We use an asterisk for complex conjugation.

function $G_k(t, s, \lambda)=G_k(s, t, \lambda)=\alpha_k(t, \lambda)\beta_k(s, \lambda)$ if $t\geqq s$.

If $\lambda>0$, then $W^*_{k,u}(x)=W_{k,-u}(x)=W_{k,u}(x)$ and $\beta_k(x, \lambda)-\beta^*_k(x, \lambda)$ $=i\pi^{-1}\sinh(2\pi\lambda^{1/2})|\Gamma(1/2-k-u)|^2\alpha_k(x, \lambda)$. Thus if $t\geqq s$ it follows that

$$\begin{aligned}\operatorname{Im} G_k(t, s, \lambda) &= -2^{-1}i[\beta_k(s, \lambda)-\overset{*}{\beta_k}(s, \lambda)]\alpha_k(t, \lambda)\\ &= (2\pi)^{-1}\sinh(2\pi\lambda^{1/2})\,|\,\Gamma(1/2-k-u)\,|^2\alpha_k(s, \lambda)\alpha_k(t, \lambda).\end{aligned}$$

For fixed s and t, G_k is meromorphic in $\operatorname{Re}\lambda<0$, and the poles of G_k in the λ plane are determined by the poles of $\Gamma(1/2-k-u)$ Thus if $k<1/2$, then G_k has no poles. Suppose $k\geqq 1/2$. We put $\lambda_{n,k}=-(k-1/2-n)^2$, $n=0, 1, 2, \cdots, N_k$, where $N_k\leqq k-1/2$. The residue of $\Gamma(1/2-k-u)$ at $\lambda_{n,k}$ is $(-1)^n(n!)^{-1}(2n+1-2k)$, so the residue of G_k at $\lambda_{n,k}$ for $t\geqq s$ is

$$\begin{aligned}&\frac{(-1)^n(2n+1-2k)}{n!\Gamma(2n-2k)}\beta_k(s, \lambda_{n,k})\cdot\alpha_k(t, \lambda_{n,k})\\ &\qquad\qquad= \frac{2n+1-2k}{n!\Gamma(2k-n)}\alpha_k(s, \lambda_{n,k})\alpha_k(t, \lambda_{n,k}).\end{aligned}$$

Hence, by **[11; 4]** we have established

THEOREM 1. *Suppose* $-\infty<k<\infty$. *Let* $\rho_k(\lambda)$ *be the monotone increasing function* $=(1/2\pi^2)\int_0^\lambda \sinh(2\pi\xi^{1/2})|\Gamma(1/2-k-i\xi^{1/2})|^2 d\xi$ *if* $\lambda\geqq 0$, $=0$ *if* $\lambda<0$, $k<1/2$, *and* $=\sum_{\lambda<\lambda_{n,k}}(2n+1-2k)/n!\Gamma(2k-n)$ *if* $\lambda<0$, $k\geqq 1/2$. *By* $L^2(d\rho_k)$ *we mean the Hilbert space with norm given by* $\|g\|=[\int_{-\infty}^{\infty}|g(\lambda)|^2 d\rho_k(\lambda)]^{1/2}$.

Let U_k *be the operator on* $L^2(0, \infty)$ *to* $L^2(d\rho_k)$ *defined by* $(U_kf)(\lambda)$ $=\text{l.i.m.}_{\epsilon\to 0+}\int_\epsilon^\infty\alpha_k(x, \lambda)f(x)dx$, $f\in L^2(0, \infty)$, $-\infty<\lambda<\infty$. *Then*

(i) U_k *is an isometric transformation that maps* $L^2(0, \infty)$ *onto* $L^2(d\rho_k)$, *so if* $f\in L^2(0, \infty)$ *and* $g=U_kf$, *then*

$$\begin{aligned}\int_0^\infty |f(x)|^2dx &= \int_{-\infty}^\infty |g(\lambda)|^2 d\rho_k(\lambda)\\ &= \frac{1}{2}\pi^{-2}\int_0^\infty |g(\lambda)|^2\sinh(2\pi\lambda^{1/2})\left|\Gamma\left(\frac{1}{2}-k-i\lambda^{1/2}\right)\right|^2 d\lambda\\ &\quad+\sum_{n=0}^{N_k}|g(\lambda_{n,k})|^2\frac{2k-2n-1}{n!\Gamma(2k-n)}.\end{aligned}$$

(ii) *For any* $g\in L^2(d\rho_k)$, $(U_k^{-1}g)(x)=\int_{-\infty}^\infty\alpha_k(x, \lambda)g(\lambda)d\rho_k(\lambda)$, *where the integral is understood to converge in* $L^2(d\rho_k)$ *norm.*

(iii) *If* $\lambda g(\lambda)\in L^2(d\rho_k)$, *then* $(U_kL_kU_k^{-1}g)(\lambda)=\lambda g(\lambda)$ *except for a set of* $d\rho_k$ *measure* 0.

3. Thus the isometric map U_k diagonalizes L_k. Next we consider a class of integral operators on $L^2(0, \infty)$ that are bounded functions of L_k.

THEOREM 2. *Suppose* Re $\gamma>0$ *and* $1/2-k+\gamma\neq 0, -1, -2, \cdots$. *Let* $\mathcal{K}_{k,\gamma}$ *be the operator on* $L^2(0, \infty)$ *to* $L^2(0, \infty)$ *defined by*

$$(\mathcal{K}_{k,\gamma}f)(x) = \Gamma(1/2 - k + \gamma)\int_0^\infty (xt)^{\gamma-1/2}(x + t)^{-\gamma-1/2}W_{k,\gamma}(x + t)f(t)dt.$$

Then $\mathcal{K}_{k,\gamma}$ *is a bounded normal operator such that*

$$(U_k\mathcal{K}_{k,\gamma}U_k^{-1}g)(\lambda) = \Gamma(\gamma + i\lambda^{1/2})\Gamma(\gamma - i\lambda^{1/2})g(\lambda)$$

except for a set of $d\rho_k$ *measure* 0.

PROOF. Hari Shanker [10] showed that if Re $(\gamma\pm u)>0$, $1/2-k+\gamma\neq 0, -1, -2, \cdots$, then $\Gamma(\gamma + u)\Gamma(\gamma - u)W_{k,u}(x)x^{-1}=\Gamma(1/2+\gamma-k)\int_0^\infty(xt)^{\gamma-1/2}(x+t)^{-\gamma-1/2}W_{k,\gamma}(x+t)W_{k,u}(t)t^{-1}dt$. If $k>1/2$, Re $\gamma>0$, $u_n=i\lambda_{n,k}^{1/2}$, $n=0, 1, \cdots, N_k$, $N_k<k-1/2$, then $W_{k,u_n}(t)t^{\gamma-3/2}\in L(0, \infty)$, so in this case the condition Re $(\gamma\pm u)>0$ may be replaced by the restriction Re $\gamma>0$. Thus if $g(\lambda)$ is continuous with compact support the Fubini theorem assures us that

$$(\mathcal{K}_{k,\gamma}U_k^{-1}g)(x) = \Gamma(\gamma - i\lambda^{1/2})\Gamma(\gamma - i\lambda^{1/2})(U_k^{-1}g)(x).$$

By operating on the left with U_k we obtain

$$(*)\qquad (U_k\mathcal{K}_{k,\gamma}U_k^{-1}g)(\lambda) = \Gamma(\gamma + i\lambda^{1/2})\Gamma(\gamma - i\lambda^{1/2})g(\lambda).$$

Since $\Gamma(\gamma+i\lambda^{1/2})\Gamma(\gamma-i\lambda^{1/2})$ is a.e. bounded with respect to $d\rho_k$ measure, $\mathcal{K}_{k,\gamma}$ is a bounded operator. Finally, (*) holds for all g in a dense subset of $L^2(d\rho_k)$ and hence for all $g\in L^2(d\rho_k)$.

By specializing to $k=0$ we obtain a result of Lebedev [5; 6].

COROLLARY 3 (LEBEDEV).

$$(U_0f)(\lambda) = \operatorname*{l.i.m.}_{\epsilon\to 0+} \pi^{-1/2}\int_\epsilon^\infty K_{i\lambda^{1/2}}(x/2)x^{-1/2}f(x)dx$$

provides an isometric map of $L^2(0, \infty)$ *onto the Hilbert space with norm given by* $\|g\|=[\int_0^\infty|g(x)|^2\sinh(\pi\lambda^{1/2})d\lambda]^{1/2}$, *with*[3]

$$(U_0^{-1}g)(x) = \pi^{-3/2}\int_0^\infty K_{i\lambda^{1/2}}(x)x^{-1/2}g(\lambda)\sinh(\pi\lambda^{1/2})d\lambda.$$

[3] K_u is the modified Bessel function of the third kind.

Let $(\mathcal{H}_{0,1/2} f)(x) = \int_0^\infty e^{-(x+y)/2}(x+y)^{-1}f(y)dy$. *Then* $(U_0\mathcal{H}_{0,1/2}\, U_0^{-1}g)(\lambda) = \pi \operatorname{sech}(\pi\lambda^{1/2})g(\lambda) = \pi \operatorname{sech}(\pi L^{1/2})g(\lambda)$.

PROOF. Use $W_{0,u}(x) = \pi^{-1/2}x^{1/2}K_u(x/2)$, $W_{0,1/2}(x) = e^{-x/2}$ and $\Gamma(1/2 - i\lambda^{1/2})\Gamma(1/2+i\lambda^{1/2}) = \pi \operatorname{sech}(\pi\lambda^{1/2})$.

THEOREM 4. *Let* $\phi_n(x) = e^{-x/2}L_n(x)$, $n = 0, 1, 2, \cdots$, *where* L_n *is the nth Laguerre function. Define the operator* V_k *on* l^2 *by specifying that whenever* $a = \{a_n\}_0^\infty \in l^2$, *then* $(V_k a)(\lambda) = U_k(\sum_{n=0}^\infty a_n\phi_n)$. *It follows that*:

(i) V_k *is an isometric map of* l^2 *onto* $L^2(d\rho_k)$ *whose inverse* V_k^{-1} *is given by* $V_k^{-1}g = a = \{a_n\}$, *where*

$$a_n = \int_0^\infty (U_k^{-1}g)(x)\phi_n(x)dx, \qquad n = 0, 1, 2, \cdots.$$

(ii) *If* $g \in L^2(d\rho_k)$, *and k is not a positive integer, then* $(V_k H_k V_k^{-1}g)(\lambda) = (U_k\mathcal{H}_{k,1/2}U_k^{-1}g)(\lambda) = \pi \operatorname{sech}(\pi\lambda^{1/2})g(\lambda)$ *except for a set of* $d\rho_k$ *measure zero.*

PROOF. (i) is true since U_k is isometric and the ϕ_n form a complete orthonormal set in $L^2(0, \infty)$. (ii) is a consequence of the relation $\int_0^\infty (\mathcal{H}_{k,1/2}\phi_n)(x)\phi_m(x)dx = (n+m+1-k)^{-1}$, $n, m = 0, 1, 2, \cdots$, proved in [9] for $k<1$ and easily seen valid for all $k \neq 1, 2, 3, \cdots$ by an analytic continuation argument.

Thus the Hilbert matrix H_k has the same spectrum as the multiplication operator $\pi \operatorname{sech}(\pi\lambda^{1/2})$ on $L^2(d\rho_k)$, and we have our

THEOREM 5.

(i) *For all real* $k \neq 1, 2, \cdots$, H_k *has continuous spectra of multiplicity one on* $[0, \pi]$;

(ii) *If* $k \leqq 1/2$, H_k *has no point spectrum*;

(iii) *If* $k > 1/2$, *let p and q be the largest non-negative integers such that* $2p < k - 1/2$ *and* $2q < k - 3/2$ *respectively. Then* $\pi \csc \pi k$ *and* $-\pi \csc \pi k$ *are eigenvalues of* H_k *of multiplicities* $p+1$ *and* $q+1$ *respectively.* H_k *has no other point spectrum.*

PROOF. The closure of the range of $\pi \operatorname{sech}(\pi\lambda^{1/2})$, $0 \leqq \lambda < \infty$ is $[0, \pi]$ so (i) is proved. (iii) follows from an examination of $\pi \operatorname{sech}(\pi\lambda^{1/2})$, $\lambda^{1/2} = i(k - 1/2 - n)$, $n = 0, 1, \cdots, N_k$, $N_k < k - 1/2$.

The eigenvalues and corresponding eigenvectors in (iii) were exhibited by Hill [2]. Theorem 5 provides a complete determination of the spectrum of H_k and thus solves a problem posed by Magnus in [7].

References

1. A. Erdélyi, *Higher transcendental functions*, I, McGraw-Hill, 1953.

2. C. K. Hill, *The Hilbert bound of a certain doubly-infinite matrix*, J. London Math. Soc. vol. 32 (1957) pp. 7–17.

3. A. E. Ingham, *A note on Hilbert's inequality*, J. London Math. Soc. vol. 11 (1936) pp. 237–240.

4. K. Kodaira, *The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices*, Amer. J. Math. vol. 71 (1949) pp. 921–945.

5. N. N. Lebedev, *Some singular integral equations connected with integral representations of mathematical physics*, C. R. (Doklady) Acad. Sci. URSS. vol. 65 (1949) pp. 621–624.

6. ———, *The analogue of Parseval's theorem for a certain integral transform*, C. R. (Doklady) Acad. Sci. URSS. vol. 68 (1949) pp. 653–656.

7. W. Magnus, *Ueber einige beschränkte Matrizen*, Archiv der Mathematik, vol. 2 (1949–1950) pp. 405–412.

8. ———, *On the spectrum of Hilbert's matrix*, Amer. J. Math. vol. 72 (1950) pp. 699–704.

9. M. Rosenblum, *On the Hilbert matrix* I, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 137–140.

10. H. Shanker, *An integral equation for Whittaker's confluent hypergeometric function*, Proc. Cambridge Philos. Soc. vol. 45 (1949) pp. 482–483.

11. E. C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations*, Oxford, 1946.

University of Virginia