BASIC SETS OF POLYNOMIAL SOLUTIONS FOR PARTIAL DIFFERENTIAL EQUATIONS

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1. In this note I present an algebraic method for constructing basic sets of polynomials which are solutions of a linear homogeneous partial differential equation with constant coefficients. This method generalizes and unifies several known results (see §3).

Let $E=R^n$ $(n>1)$ be the euclidean space of dimension n, whose points shall be $x=(x_1, \cdots, x_n)$. The capital letters M and J will denote multi-indices $M=(m_1, \cdots, m_n)$, $J=(j_1, \cdots, j_n)$, where the m_i and j_i are positive integers; the corresponding lower-case letters will mean $m=|M|=m_1+\cdots+m_n$. We shall also write $x^J=x_1^{j_1}\cdots x_n^{j_n}$.

We consider a linear homogeneous partial differential operator with constant coefficients of order m of the form

$$D = \sum_{|M|=m} \alpha_M D^M \tag{1}$$

where

$$D^M = \frac{\partial^m}{\partial x^M} = \frac{\partial^m}{\partial x_1^{m_1}\cdots \partial x_n^{m_n}}\,.$$

Let $\vee E$ be the symmetrical algebra of E, direct sum of the symmetrical powers $\vee^j E$ [1, §1, Exercises 1–2, p. 15]. We identify $\vee^1 E$ with E and $\vee^0 E$ with R. The vector space $\vee^j E$ has dimension $C_{n+j-1,j}$ over R (see §2) and has a basis formed by all products

$$\overset{J}{e} = \overset{j_1}{e_1}\cdots \overset{j_n}{e_n} \tag{2}$$

with $|J|=j$, where $e_1, \cdots, e_n$ is the canonical basis of E.

Consider the element

$$a = \sum_{|M|=m} \alpha_M e^M \in \vee^m E \tag{3}$$

and let $\mathfrak{a}$ be the ideal of $\vee E$ generated by a. Let $A_j=\mathfrak{a}\cap \vee^j E$ be the j-th homogeneous component of $\mathfrak{a}$; clearly $A_j=\{0\}$ for $j<m$. For $x\in E$ let the element $x\vee\cdots\vee x$ (j factors) of $\vee^j E$ be written as x^j (this is not to be confused with x^J, which is a scalar). We have

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$$Dx^j \in A_j. \tag{4}$$

In fact from

$$\frac{\partial}{\partial x_i} x^j = j e_i x^{j-1}$$

it follows that

$$Dx^j = j(j-1) \cdots (j-m+1) a x^{j-m} \in A_j.$$

2. Consider now the quotient algebra $Q = \bigvee E/\mathfrak{a}$, which is a graded algebra whose homogeneous components are the vector spaces $\bigvee^j E/A_j$. Let θ be the canonical homomorphism of $\bigvee E$ onto Q, then it follows from (4) that the components of $\theta(x^j)$ with respect to a given basis of $\bigvee^j E/A_j$ are homogeneous polynomials Y_j of degree j in $x_1, \cdots, x_n$ which satisfy the algebraic relation $DY_j = 0$.

In particular let Q_j be a supplementary subspace to A_j in $\bigvee^j E$. Then Q_j is canonically isomorphic to $\bigvee^j E/A_j$, with which we identify it, and θ becomes the projection of $\bigvee^j E$ onto Q_j, parallel to A_j.

Suppose that the coefficient $\alpha_{M^0} = \alpha_{(m_1^0, \cdots, m_n^0)}$ in (1) is different from zero and take for Q_j the subspace spanned by all the products e^J where at least one of the relations $j_1 < m_1^0, \cdots, j_n < m_n^0$ is satisfied. These products are linearly independent modulo A_j and their number (i.e. the dimension of Q_j) is

$$\binom{n+j-1}{j} - \binom{n+j-m-1}{j-m}. \tag{5}$$

Indeed, the number of all the solutions of the equation

$$j_1 + \cdots + j_n = j \tag{6}$$

in positive integers $j_1, \cdots, j_n$ is $C_{n+j-1,j}$ and the number of those solutions of (6) which verify all the relations $j_1 \geqq m_1^0, \cdots, j_n \geqq m_n^0$ is the same as the number of all the solutions of

$$\xi_1 + \cdots + \xi_n = j - m$$

in positive integers $\xi_1, \cdots, \xi_n$, i.e. $C_{n+j-m-1,j-m}$. Thus the number of those solutions of (6) for which at least one relation $j_i < m_i^0$ holds, is the difference (5).

The components Y_j^J of $\theta(x^j)$ with respect to the basis e^J of Q_j are linearly independent, since every one of them contains exactly one term x^J in which at least one j_i verifies $j_i < m_i^0$ and no two different polynomials Y_j^J contain the same term of this type. On the other hand there are at most (5) linearly independent homogeneous polynomials Y_j of degree j which satisfy $DY_j=0$, since there are altogether $C_{n+j-1,j}$ linearly independent homogeneous polynomials of degree j and $DY_j=0$ gives $C_{n+j-m-1,j-m}$ linear relations among the coefficients of Y_j. These relations can be seen to be independent if we order D lexicographically according to M and DY_j according to the exponents of the x_i [**8**, Footnote p. 428].

Let us observe finally that the relation $\theta(x^{j+k})=\theta(x^j)\theta(x^k)$ yields recurrence formulas between the Y_j^J.

3. **Examples.** (1) Consider the operator

$$\frac{\partial^m}{\partial x_1^m}+\cdots+\frac{\partial^m}{\partial x_n^m}\cdot$$

The element a of (3) is now

$$\overset{m}{e_1}+\cdots+\overset{m}{e_n}$$

and a basis of Q_j is formed by all products e^J of (2) with $j_n<m$. To calculate the components of $\theta(x^j)\in Q_j$ with respect to this basis we develop x^j according to the polynomial theorem and reduce every term in which an e^J occurs with $j_n\geqq m$ using the relation

$$\overset{m}{e_n}=-\overset{m}{e_1}-\cdots-\overset{m}{e_{n-1}}$$

(see [**2**, pp. 56–58], where the detailed calculation is carried out for the case $m=2$). The coefficients of $\theta(x^j)$ with respect to our basis (e^J) will then be the polynomials of Miles-Williams [**5**; **6**; **8**]:

$$(7)\qquad Y_j^J(x)=\sum(-1)^{[\mu_n/m]}\frac{j!}{\prod_{i=1}^{n}\mu_i!}\,\frac{[\mu_n/m]!}{\prod_{i=1}^{n-1}\left(\frac{j_i-\mu_i}{m}\right)!}\,x_1^{\mu_1}\cdots x_n^{\mu_n}$$

where the summation extends over all systems $\mu_1,\cdots,\mu_n$ such that

$$\mu_i\equiv j_i \pmod m\qquad i=1,2,\cdots,n-1,$$
$$\sum_{i=1}^{n}\mu_i=j,$$
$$\mu_i\leqq j_i,\qquad i=1,2,\cdots,n-1.$$

It is evident from the above construction that for $n=2$, $m=2$, the polynomials are the real and imaginary parts of $(x_1+ix_2)^j$ [**6**].

The present method for obtaining the polynomials (7) in the case $m=2$ figures in my paper [**2**], where it is used to calculate the Fourier transform of $Y_j(x)\cdot|x|^{-n}$, where $Y_j(x)$ is a homogeneous harmonic polynomial of degree j. At that time I had no knowledge of the work of Miles and Williams, but the present article grew out of an effort to obtain a noncomputational proof of their results [**5**; **6**; **7**; **8**; **9**] and to extend them.

A very similar construction to the present one has been given by Protter [**10**] in the case $n=3$. He obtains all the powers x^j at once by considering the function $\exp x=\sum x^j/j!$. Still another similar construction figures in an earlier paper of Whittaker [**11**].

(2) For the wave operator

$$\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_{n-1}^2}-\frac{\partial^2}{\partial x_n^2}$$

the basis of Q_j is also formed by the e^J with $j_n<2$, but for the expression of $\theta(x^j)\in Q_j$ in terms of this basis the relation

$$e_n^2=e_1^2+\cdots+e_{n-1}^2$$

is used. The polynomials obtained are again those of Miles and Williams [**5**] and differ from (7) in the absence of the factor $(-1)^{[\mu_n/2]}$.

(3) Consider the iterated Laplacian for the case[1] $n=2$:

$$\Delta^2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2.$$

The element a of (3) is now

$$(e_1^2+e_2^2)^2=e_1^4+2e_1^2e_2^2+e_2^4.$$

A basis of Q_j is given by

$$e_1^j,\ e_1^{j-1}e_2,\ e_1^{j-2}e_2^2,\ e_1^{j-3}e_2^3 \tag{8}$$

and we have the relation

$$e_2^4=-e_1^4-2e_1^2e_2^2 \tag{9}$$

and more generally

$$e_2^{4t}=-(2t-1)e_1^{4t}-2te_1^{4t-2}e_2^2$$

which can be proved by mathematical induction. This last relation yields[2]

[1] We shall write x, y instead of x_1, x_2.

[2] We shall write simply x^j instead of $\theta(x^j)$ in the sequel.

$$
\begin{aligned}
(xe_1 + ye_2)^j &= \sum_{s=0}^{j} \binom{j}{s} x^{j-s} y^s e_1^{j-s} e_2^s \\
&= x^j e_1^j + jx^{j-1} y e_1^{j-1} e_2 + \binom{j}{2} x^{j-2} y^2 e_1^{j-2} e_2^2 + \binom{j}{3} x^{j-3} y^3 e_1^{j-3} e_2^3 \\
&\quad - \sum_{t=1}^{[j/4]} \binom{j}{4t} x^{j-4t} y^{4t} e_1^{j-4t} \{(2t-1)e_1^{4t} + 2te_1^{4t-2} e_2^2\} \\
&\quad - \sum_{t=1}^{[(j-1)/4]} \binom{j}{4t+1} x^{j-4t-1} y^{4t+1} e_1^{j-4t-1} e_2 \{(2t-1)e_1^{4t} + 2te_1^{4t-2} e_2^2\} \\
&\quad - \sum_{t=1}^{[(j-2)/4]} \binom{j}{4t+2} x^{j-4t-2} y^{4t+2} e_1^{j-4t-2} e_2^2 \{(2t-1)e_1^{4t} + 2te_1^{4t-2} e_2^2\} \\
&\quad - \sum_{t=1}^{[(j-3)/4]} \binom{j}{4t+3} x^{j-4t-3} y^{4t+3} e_1^{j-4t-3} e_2^3 \{(2t-1)e_1^{4t} + 2te_1^{4t-2} e_2^2\} .
\end{aligned}
$$

Collecting terms with the help of (9) and of

$$
e_2^5 = - e_1^4 e_2 - 2e_1^3 e_2^2,
$$

we obtain the four homogeneous biharmonic polynomials of degree j

$$
\begin{aligned}
Y_j^{(j,0)} &= \sum_{\mu=0}^{[j/2]} (-1)^{\mu-1} (\mu - 1) \binom{j}{2\mu} x^{j-2\mu} y^{2\mu}, \\
Y_j^{(j-1,1)} &= \sum_{\mu=0}^{[(j-1)/2]} (-1)^{\mu-1} (\mu - 1) \binom{j}{2\mu+1} x^{j-2\mu-1} y^{2\mu+1}, \\
Y_j^{(j-2,2)} &= \sum_{\mu=0}^{[j/2]} (-1)^{\mu-1} \mu \binom{j}{2\mu} x^{j-2\mu} y^{2\mu}, \\
Y_j^{(j-3,3)} &= \sum_{\mu=0}^{[(j-1)/2]} (-1)^{\mu-1} \mu \binom{j}{2\mu+1} x^{j-2\mu-1} y^{2\mu+1},
\end{aligned}
\tag{10}
$$

which are the coefficients of the four elements (8), respectively. These biharmonics are different from those of Miles and Williams [**9**], but are closely related to them.

It is very easy to obtain recurrence relations for the polynomials (10). Comparing

$$
(xe_1 + ye_2)^{j+1} = \sum_{\nu=0}^{3} Y_{j+1}^{(j+1-\nu,\nu)} e_1^{j+1-\nu} e_2^{\nu}
$$

with

$$(xe_1 + ye_2)(xe_1 + ye_2)^{j} = (xe_1 + ye_2) \cdot \sum_{\nu=0}^{3} Y_j^{(j-\nu,\nu)} e_1^{j-\nu} e_2^{\nu}$$

and using (9), we obtain for $j \geqq 3$,

$$\begin{aligned} Y_{j+1}^{(j+1,0)} &= xY_j^{(j,0)} - yY_j^{(j-3,3)}, \\ Y_{j+1}^{(j,1)} &= xY_j^{(j-1,1)} + yY_j^{(j,0)}, \\ Y_{j+1}^{(j-1,2)} &= xY_j^{(j-2,2)} + yY_j^{(j-1,1)} - 2yY_j^{(j-3,3)}, \\ Y_{j+1}^{(j-2,3)} &= xY_j^{(j-3,3)} + yY_j^{(j-2,2)}. \end{aligned}$$

Analogous recurrence relations for the Miles-Williams biharmonics have been established by Wicht [12].

We could treat in a similar way the k times iterated Laplacian ($m = 2k$) in n variables.

(4) Let us consider the operator

$$q \frac{\partial^3}{\partial x^3} + p \frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial y^3} \cdot$$

The basis of Q_j is now

$$e_1^{j}, \; e_1^{j-1} e_2, \; e_1^{j-2} e_2^{2}$$

and to find the components of $x^j \in Q_j$ we use the relation

$$e_2^{3} = -qe_1^{3} - pe_1^{2}e_2. \tag{11}$$

The homogeneous solutions $u_j(x, y)$, $v_j(x, y)$, $w_j(x, y)$ of degree j are defined by

$$(xe_1 + ye_2)^{j} = u_j(x, y)e_1^{j} + v_j(x, y)e_1^{j-1} e_2 + w_j(x, y)e_1^{j-2} e_2^{2}.$$

Comparing

$$(xe_1 + ye_2)^{j+k} = u_{j+k}e_1^{j+k} + v_{j+k}e_1^{j+k-1} e_2 + w_{j+k}e_1^{j+k-2} e_2^{2}$$

with

$$\begin{aligned} &(xe_1 + ye_2)^{j}(xe_1 + ye_2)^{k} \\ &\qquad = (u_je_1^{j} + v_je_1^{j-1} e_2 + w_je_1^{j-2} e_2^{2})(u_ke_1^{k} + v_ke_1^{k-1} e_2 + w_ke_1^{k-2} e_2^{2}) \end{aligned}$$

and using (11) we obtain

$$\begin{aligned} u_{j+k} &= u_j u_k - q(v_j w_k + w_j v_k), \\ v_{j+k} &= u_j v_k + v_j u_k - p(v_j w_k + w_j v_k) - q w_j w_k, \\ w_{j+k} &= u_j w_k + v_j v_k + w_j u_k - p w_j w_k. \end{aligned}$$

These relations are due to Lammel [**3**; **4**, p. 194].

(5) Consider finally the Cauchy-Riemann operator

$$2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \cdot \tag{12}$$

We have now $e_1 + ie_2 = 0$, every Q_j has dimension 1, basis e_1^j, and

$$(xe_1 + ye_2)^j = (xe_1 + yie_1)^j = (x + iy)^j e_1^j.$$

The homogeneous polynomials corresponding to (12) are

$$(x + iy)^j = z^j.$$

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