ON THE LATTICE OF ALL JOIN-ENDOMORPHISMS OF A LATTICE

G. GRÄTZER AND E. T. SCHMIDT

A many-one correspondence Θ of the lattice L into itself is called a join-endomorphism, if it satisfies

$$(1) \quad \Theta(x \cup y) = \Theta x \cup \Theta y$$

for all $x, y \in L$.

It is easily shown that every join-endomorphism is an isotone correspondence (i.e. $x \geq y$ implies $\Theta x \geq \Theta y$). It is also easy to see that the set I of antecedents of 0 under any join-endomorphism is an ideal. (For these and other facts used in the sequel we refer to the textbook of G. Birkhoff, *Lattice theory*, rev. ed., New York, 1948, henceforth cited as LT.)

G. Birkhoff states in LT (p. 208, Example 4) that all join-endomorphisms Θ of any lattice L form an l -semigroup, where the join of two join-endomorphisms Θ and Φ satisfies

$$(2) \quad (\Theta \cup \Phi)x = \Theta x \cup \Phi x$$

for all $x \in L$.

But this statement is not true in general; indeed, we shall show that there exists a lattice whose join-endomorphisms do not form an l -semigroup under the join stated.

We shall also deal with the following question proposed by G. Birkhoff in LT.

Problem 93. Is the lattice of all join-endomorphisms of an arbitrary lattice semi-modular?

We shall show that the answer is negative.¹ Namely, restricting ourselves to finite lattices, we shall prove that there exists no lattice whose lattice of all join-endomorphisms is semi-modular and not distributive; furthermore, the lattice of all join-endomorphisms of any lattice L is distributive if and only if L is distributive.

We give also some generalizations of these results.

1. On the existence of the lattice of all join-endomorphisms. In what follows $L_{\cup}$ will denote the set of all join-endomorphisms of a lattice L , where the join of two elements Θ and Φ of $L_{\cup}$ is defined by

Received by the editors July 12, 1957; and, in revised form, November 22, 1957.

¹ This much was remarked by R. P. Dilworth in his review of LT, Bull. Amer. Math. Soc. vol. 56 (1950) pp. 204–206.

(2); the join operation gives rise to a partial ordering in $L_{\cup}$; $\Theta \leq \Phi$ if and only if $\Theta x \leq \Phi x$ for all $x \in L$.

Firstly we show by an example that $L_{\cup}$ is in general not a lattice.

Let V be the chain of all real numbers of the closed interval $[0, 1]$ with the usual ordering. Let us consider in $V \cdot V$ (the cardinal product of V by itself, in the sense of LT p. 7) the sublattice L consisting of all elements of $V \cdot V$, with the exception of $(1, 0)$. L is actually a sublattice of $V \cdot V$, for $(1, 0)$ is join- and meet-irreducible.

Let us consider the following mappings of L into itself:

$$\begin{aligned}\Theta &= \{(1, y) \rightarrow (1, 1) \text{ if } y \neq 0; (x, y) \rightarrow (0, 0) \text{ if } x \neq 1\}, \\ \Phi &= \{(x, 0) \rightarrow (0, 0) \text{ if } x \neq 1; (x, y) \rightarrow (1, y) \text{ if } y \neq 0\}.\end{aligned}$$

It is routine to check that $\Theta, \Phi \in L_{\cup}$. For all $\psi \in L_{\cup}$ satisfying $\psi \leq \Theta$ and $\psi \leq \Phi$ we have $\psi(x, y) \leq \Theta(x, y) = (0, 0)$ if $x \neq 1$, and for this reason $\psi(1, y) = \psi(1, z)$ holds for all $y \neq 0$ and $z \neq 0$. Let $\psi(1, y) = (a, b)(y \neq 0)$. From $\Phi(1, y) = (1, y)$ and $\psi \leq \Phi$ it follows that $(a, b) \leq (1, y)$ for all $y \neq 0$, that is $b = 0$. Thus each ψ is of the form $\psi = \{(x, y) \rightarrow (0, 0) \text{ if } x \neq 1; (1, y) \rightarrow (a, 0) \text{ if } y \neq 0\}$. Since among these ψ there is clearly no greatest one, $a = 1$ being impossible, it follows that $L_{\cup}$ is not a lattice.

In what follows we shall need the following sufficient condition for $L_{\cup}$ to be a lattice.

THEOREM 1. *If L is a complete lattice, then $L_{\cup}$ is also a complete lattice.*

PROOF. Let $\Theta_{\alpha} \in L_{\cup}$ ($\alpha \in A$) and define Θ by $\Theta x = \bigvee_{\alpha \in A} \Theta_{\alpha} x$. From the general associative law we get

$$\begin{aligned}\Theta x \cup \Theta y &= \bigvee_{\alpha \in A} \Theta_{\alpha} x \cup \bigvee_{\alpha \in A} \Theta_{\alpha} y = \bigvee_{\alpha \in A} (\Theta_{\alpha} x \cup \Theta_{\alpha} y) \\ &= \bigvee_{\alpha \in A} \Theta_{\alpha} (x \cup y) = \Theta (x \cup y)\end{aligned}$$

i.e., $\Theta \in L_{\cup}$. Clearly $\Theta \geq \Theta_{\alpha}$; moreover if $\Phi \geq \Theta_{\alpha}$ for all $\alpha \in A$, then $\Phi x \geq \bigvee_{\alpha \in A} \Theta_{\alpha} x$ whence $\Phi \geq \Theta$ and any subset of $L_{\cup}$ has a join. $L_{\cup}$ has a zero-element, for L has a 0 and the mapping $x \rightarrow 0$ for all $x \in L$ is a join-endomorphism of L . Hence $L_{\cup}$ is a partly ordered set with zero-element and complete joins, consequently $L_{\cup}$ is a complete lattice (LT, p. 49).

COROLLARY (ZACHER'S THEOREM).² *If L is a finite lattice then all*

² Giovanni Zacher, *Sugli eniormorfismi superiori ed inferiori*, Atti del Quarto Congresso dell'Unione Matematica Italiano, Taormina, vol. 2 (1951) pp. 251-252, and by the same author, with the same title in Rend. Accad. Sci. Fis. Mat. Napoli (4) vol. 19 (1952) pp. 45-56 (1953).

join-endomorphisms of L form a lattice.

2. Nondistributive lattices with finite bounded chains.³ If the lattice L is not distributive, then it contains as a sublattice one of the lattices

S : the elements of S include a, b, c, i, o ; $a \cup b = b \cup c = c \cup a = i$,
 $a \cap b = b \cap c = c \cap a = o$;

T : the elements of T include a, b, c, i, o ; $a \cup c = b \cup c = i$, $a \cap c = b \cap c = o$, $a \cap b = a$.

a. Let L be a modular, but not distributive lattice with finite bounded chains. Then L contains the lattice S as a sublattice with the further condition that a, b, c cover o (LT, p. 134).

Let us consider in L the following join-endomorphisms:

$$\Theta = \{[a] \rightarrow o; L - [a] \rightarrow a\},$$

$$\Phi = \{[o] \rightarrow o; [b] - [o] \rightarrow b; [c] - [o] \rightarrow c; L - [b] - [c] \rightarrow i\},$$

$$\Psi = \{[o] \rightarrow o; [b] - [o] \rightarrow b; L - [b] \rightarrow i\},$$

$$\Omega = \{[i] \rightarrow o; L - [i] \rightarrow a\}.$$

Then Θ covers Ω and $\Theta \cup \Phi = \{[o] \rightarrow o; L - [o] \rightarrow i\}$. Clearly $\Omega \cup \Phi = \Phi < \Psi < \Theta \cup \Phi$ so that $L_{\cup}$ is not semi-modular.

b. If L is a nonmodular lattice with finite bounded chains, then L contains as a sublattice the lattice T . It is clear that there exist in L an x and y , such that x covers $x \cap y$ and $x \cap y \neq y$. The mappings ($\mathfrak{F}$ denotes a maximal ideal, such that $c \in \mathfrak{F}$, $a \notin \mathfrak{F}$)

$$\Theta = \{\mathfrak{F} \rightarrow x \cap y; L - \mathfrak{F} \rightarrow x\},$$

$$\Phi = \{[o] \rightarrow x \cap y; [c] - [o] \rightarrow x; [b] - [o] \rightarrow y; L - [c] - [b] \rightarrow x \cup y\},$$

$$\Psi = \{[o] \rightarrow x \cap y; [c] - [o] \rightarrow x; [a] - [o] \rightarrow y;$$

$$L - [c] - [a] \rightarrow x \cup y\},$$

$$\Omega = \{[i] \rightarrow x \cap y; L - [i] \rightarrow x\}$$

are join-endomorphisms and it may be easily checked that Θ covers Ω , yet $\Theta \cup \Phi > \Psi > \Omega \cup \Phi = \Phi$, i.e. $L_{\cup}$ is not semi-modular. Thus we have the following

THEOREM 2. *The lattice of all join-endomorphisms of a nondistributive lattice with finite bounded chains is not semi-modular.*

3. The case of finite distributive lattices. Let L be a distributive lattice with 0 and 1, of finite length. It is known that L is itself finite

³ By a bounded chain we mean a chain with a least and a greatest element. $[a]$ denotes the ideal generated by a .

and in L every element is the join of join-irreducible elements. Conversely, if L has exactly k join-irreducible elements $a_1, a_2, \dots, a_k$ then $\Theta \in L_U$ is completely determined by the elements $b_i = \Theta a_i$ ($i = 1, 2, \dots, k$). Thus we may write Θ in the form $\Theta = (b_1, b_2, \dots, b_k)$, where evidently $a_i \geq a_j$ implies $b_i \geq b_j$. Moreover

LEMMA 1. *If $a_1, \dots, a_k$ are all the join-irreducible elements of L , and $b_1, \dots, b_k$ are arbitrary in L , then a necessary and sufficient condition for the existence of a join-endomorphism Θ with $\Theta a_i = b_i$ ($i = 1, 2, \dots, k$) is the fulfillment of the condition: $a_i \geq a_j$ implies $b_i \geq b_j$.*

PROOF. Let us denote by $r(a)$ the set of all join-irreducible elements contained in a ; then $a = \bigcup_{a_i \in r(a)} a_i$. We need only prove the sufficiency of the condition. If the b_i satisfy the stated condition and we define Θ as $\Theta a = \bigcup_{a_i \in r(a)} \Theta a_i$, then we have to verify that $\Theta(a \cup b) = \Theta a \cup \Theta b$ for all $a, b \in L$. Only $r(a) \vee r(b) = r(a \cup b)$ need be proved, where $\vee$ means the set-theoretical union. Evidently, $r(a) \vee r(b) \subseteq r(a \cup b)$. On the other hand, if $x \in r(a \cup b)$ and $x \notin r(a), r(b)$, then $x \cap a < x$, $x \cap b < x$ and by the distributive law $x = (x \cap a) \cup (x \cap b)$ which is a contradiction; $\Theta a_i = b_i$ is obvious, completing the proof.

LEMMA 2. *Let L be a finite distributive lattice and k the number of its join-irreducible elements. Then L_U may be imbedded as a sublattice in the cardinal product of k lattices isomorphic with L .*

PROOF. Let $\Theta = (b_1, \dots, b_k)$ and $\Phi = (c_1, \dots, c_k)$ be join-endomorphisms of the lattices L . Let us consider $\Xi = (b_1 \cap c_1, \dots, b_k \cap c_k)$ and $H = (b_1 \cup c_1, \dots, b_k \cup c_k)$. These are join-endomorphisms in view of Lemma 1. Evidently, $\Xi = \Theta \cap \Phi$ and $H = \Theta \cup \Phi$. Therefore the join-endomorphisms form a lattice isomorphic to a sublattice of the cardinal product of k lattices isomorphic with L . q.e.d.

Consequently, L_U is distributive, for it is a sublattice of a distributive lattice.

On the other hand, if L_U is distributive, then so is L , because L may be imbedded in L_U by identifying $a \in L$ with the join-endomorphism $\Theta = (a, a, \dots, a)$.

Summarizing the above statements, we arrive at

THEOREM 3. *All join-endomorphisms of a finite lattice L form a distributive lattice if and only if L is distributive.*

Obviously, our results may be generalized to lattices with 0 elements and finite bounded chains.

LEMMA 2'. Let L be a distributive lattice with 0 element and with finite bounded chains. $L_{\cup}$ may be imbedded in the discrete cardinal product of as many copies of L as there exist join-irreducible elements in L .

THEOREM 3'. Let L be a lattice with 0 element and with finite bounded chains. All join-endomorphisms of L form a distributive lattice if and only if L is distributive.

BUDAPEST, HUNGARY

LINEAR COMPLETENESS AND HYPERBOLIC TRIGONOMETRY

CURTIS M. FULTON

In this paper we show the uniqueness of the relation between a segment and its angle of parallelism as derived from a model. Upon generalizing this relation hyperbolic trigonometry follows in a remarkably simple way.

To introduce proper terminology [5, pp. 11–28] let Σ denote an *axiom system*, that is a certain set of axioms together with the undefined technical and logical or universal terms used to state the axioms. We define the terms *interpretation* and *model* in the usual fashion. It is useful to make a clear distinction between the following three concepts. (1) A Σ -statement is a meaningful expression, not necessarily true, in the technical and universal terms of Σ . (2) A T - Σ -statement is a true Σ -statement in the sense of being logically derivable from Σ . (3) If I denotes an interpretation of Σ , then an I - Σ -statement is a Σ -statement holding for the model which results from the interpretation I .

For the purpose of this paper let Σ be the postulate system of Hilbert [3, pp. 2–30] with the Euclidean axiom of parallelism replaced by the characteristic postulate of hyperbolic plane geometry [6, p. 66]. Some authors have used models to find I - Σ -statements [1, §39–117; 2]. Such a procedure, however, may be objectionable [1, §118]. Conceivably an I - Σ -statement could be made which is not a T - Σ -statement, but is merely a property of a particular model. In other words, it might be possible to find contradictory I - Σ -statements in two different models. Clearly, if this happens it indicates that our system Σ is not complete [5, pp. 33–36]. Any I - Σ -statement that is not a T - Σ -statement would still be compatible with the axioms of Σ .