ON MEANS OF ENTIRE FUNCTIONS (II)¹

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Let $f(z)$ be an entire function of order ρ and lower order λ . For δ ($0 < \delta < \infty$) and κ ($-1 \leq \kappa < \infty$), let

$$M_\delta(r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \right)^{1/\delta},$$

$$\mathfrak{M}_{\delta,\kappa}(r) = \frac{1}{r^{\kappa+1}} \int_0^r x^\kappa M_\delta(x) dx.$$

The following results can be easily verified [3, Lemmas 1 and 2].

If $M(r)$ denotes the maximum modulus of $f(z)$ for $|z| = r < R$, then

$$(1) \quad M_\delta(r) \leq M(r) \leq \left(\frac{R+r}{R-r} \right)^{1/\delta} M_\delta(R),$$

$$(2) \quad \limsup_{r \rightarrow \infty} \frac{\log \log M_\delta(r)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log \mathfrak{M}_{\delta,\kappa}(r)}{\log r} = \rho \quad (0 \leq \rho \leq \infty),$$

$$\liminf_{r \rightarrow \infty} \frac{\log \log M_\delta(r)}{\log r} = \liminf_{r \rightarrow \infty} \frac{\log \log \mathfrak{M}_{\delta,\kappa}(r)}{\log r} = \lambda \quad (0 \leq \lambda \leq \infty).$$

It is also known [1] that for a fixed δ , $\log M_\delta(r)$ is an increasing convex function of $\log r$. For a fixed r , $M_\delta(r)$ tends to $M(r)$ as δ tends to infinity.

Here we deduce the following:

THEOREM 1. *For an entire function $f(z)$ of finite order ρ*

$$\log M_\delta(r) \sim \log M(r) \quad (0 < \delta < \infty).$$

THEOREM 2. *If $0 < \lambda < \rho < \infty$ is satisfied, then*

$$\log \mathfrak{M}_{\delta,\kappa}(r) \sim \log M_\delta(r) \quad (0 < \delta < \infty; -1 \leq \kappa < \infty).$$

PROOF OF THEOREM 1. Since $\log M_\delta(r)$ is a convex function of $\log r$, we have

$$(3) \quad \log M_\delta(r) = \log M_\delta(r_0) + \int_{r_0}^r \frac{m_\delta(x)}{x} dx,$$

where $m_\delta(x)$ is a nondecreasing function of x . $f(z)$ being of finite order ρ , we have for an arbitrary $\epsilon > 0$

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$$\log M_{\delta}(r) < r^{\rho+\epsilon},$$

and therefore according to (3)

$$\int_r^{2r} \frac{m_{\delta}(x)}{x} dx < (2r)^{\rho+\epsilon}$$

and since $m_{\delta}(x)$ is nondecreasing

$$m_{\delta}(r) \log 2 < (2r)^{\rho+\epsilon}.$$

ϵ being arbitrary, we get $m_{\delta}(r) < r^{\rho+\epsilon}$. From (3) we have

$$\log M_{\delta}(R) = \log M_{\delta}(r) + \int_r^R \frac{m_{\delta}(x)}{x} dx$$

where

$$\begin{aligned} \int_r^R \frac{m_{\delta}(x)}{x} dx &< m_{\delta}(R) \log \frac{R}{r} \\ &= m_{\delta}(R) \log \left(1 + \frac{R-r}{r} \right) \\ &< R^{\rho+\epsilon} \cdot \frac{R-r}{r} \\ &= r^{\rho+\epsilon} \left(1 + \frac{R-r}{r} \right)^{\rho+\epsilon} \cdot \frac{R-r}{r}. \end{aligned}$$

Now choosing R subject to the condition

$$\frac{R-r}{r} = \frac{1}{r^{\rho+\epsilon}}$$

we get

$$\int_r^R \frac{m_{\delta}(x)}{x} dx < 2.$$

With this choice of R it follows from (1), that

$$\begin{aligned} \log M_{\delta}(r) &\leq \frac{1}{\delta} \log \frac{R+r}{R-r} + \log M_{\delta}(R) \\ &< \frac{1}{\delta} \log (1 + 2r^{\rho+\epsilon}) + \log M_{\delta}(r) + 2 \\ &= (1 + o(1)) \log M_{\delta}(r). \end{aligned}$$

With $M_\delta(r) \leq M(r)$, it gives

$$\log M(r) \sim \log M_\delta(r).$$

REMARK 1. From the method of proof it is clear that the theorem $(\log M(r) \sim \log M_\delta(r))$ is also true for the class of functions of infinite order for which for some finite κ

$$\frac{l_{\kappa+1}M(r)}{\log r} = \rho_\kappa$$

is finite but $\rho_{\kappa-1}$ is infinite, $l_\kappa x$ denotes the κ th iterate of $\log x$, i.e. $l_1 x = \log x$, $l_2 x = \log \log x$, $l_3 x = \log(l_2 x)$ etc.

REMARK 2. For functions of finite order ρ and $1 \leq \rho < \infty$ the theorem can be deduced more simply as follows.

If $f(z)$ is represented by the power series $\sum_{n=0}^{\infty} a_n z^n$, then

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz$$

and we get

$$(4) \quad \mu(r) \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta = M_1(r)$$

where $\mu(r) = \max \{ |a_0|, |a_1|r, |a_2|r^2, \dots \}$. But $M_\delta(r)$ is a non-decreasing function of δ [2, pp. 14–18], therefore

$$\mu(r) \leq M_1(r) \leq M_\delta(r) \leq M(r) \quad (1 \leq \delta < \infty).$$

For finite ρ , $\log \mu(r) \sim \log M(r)$ and we get the result.

PROOF OF THEOREM 2. Under the conditions imposed, the theorem follows immediately from the results

$$\limsup_{r \rightarrow \infty} \left\{ \frac{M_\delta(r)}{\mathfrak{M}_{\delta,\kappa}(r)} \right\}^{1/\log r} = \rho, \quad \liminf_{r \rightarrow \infty} \left\{ \frac{M_\delta(r)}{\mathfrak{M}_{\delta,\kappa}(r)} \right\}^{1/\log r} = \lambda,$$

which follow on the same lines as the theorem in [3].

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