ON A CLOSURE PROPERTY OF MEASURABLE SETS[1]

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1. **Introduction.** Let X be a space. A real-valued set function $\Lambda(E)$ defined for all subsets E of X such that $0 \leqq \Lambda(E) \leqq \infty$ is called an outer measure if it satisfies the following conditions.

(i) $\Lambda(E) = 0$ if E is the empty set.

(ii) $\Lambda(E_1) \leqq \Lambda(E_2)$ if $E_1 \subset E_2$.

(iii) $\Lambda(E) \leqq \Lambda(E_1) + \Lambda(E_2) + \cdots$ if $E = E_1 \cup E_2 \cup \cdots$.

A set E is called Λ measurable if for every pair of subsets P, Q of X with $P \subset E$, $Q \subset X - E$, the equality $\Lambda(P \cup Q) = \Lambda(P) + \Lambda(Q)$ holds. Let $\mathfrak{M}(\Lambda)$ denote the family of Λ measurable sets. $\mathfrak{M}(\Lambda)$ is a completely additive class of sets and Λ is a measure on $\mathfrak{M}(\Lambda)$ (see, for example, Saks [2, pp. 45–45]). Numbers in square brackets refer to the bibliography at the end of this note. Λ is called regular if every set is contained in a Λ measurable set of equal Λ measure. Λ is called finite-valued if $\Lambda(X) < \infty$.

For each family $\mathfrak{F}$ of subsets of X let there be associated a family $\mathfrak{S}(\mathfrak{F})$ of subsets of X satisfying the following conditions. (a) $\mathfrak{F} \subset \mathfrak{S}(\mathfrak{F})$. (b) If $\mathfrak{F}_1 \subset \mathfrak{F}_2$ then $\mathfrak{S}(\mathfrak{F}_1) \subset \mathfrak{S}(\mathfrak{F}_2)$. It is the purpose of this note to establish the following closure property of measurable sets.

THEOREM. *Under the above conditions if the relationship $\mathfrak{S}[\mathfrak{M}(\Lambda)] = \mathfrak{M}(\Lambda)$ holds for every finite-valued regular outer measure Λ then it holds for every outer measure Λ.*

As an application of this theorem let $\mathfrak{S}(\mathfrak{F})$ denote the family of set obtained from $\mathfrak{F}$ by the operation (A) (see Saks [2, p. 47], for the definition of this operation). It is well known that in this case $\mathfrak{S}[\mathfrak{M}(\Lambda)] = \mathfrak{M}(\Lambda)$ for every outer measure Λ but as noted in Saks [2] the proof is much simpler if Λ is assumed to be regular (see, Kuratowski [1, p. 58]). Since $\mathfrak{S}(\mathfrak{F})$ in this case satisfies the above conditions (a) and (b), by the above theorem to show that $\mathfrak{S}[\mathfrak{M}(\Lambda)] = \mathfrak{M}(\Lambda)$ holds for every outer measure Λ it is sufficient to show that it holds for every finite-valued regular outer measure Λ.

2. **Proof of the theorem.** We first prove the following result.

LEMMA. *If Λ is an outer measure in X and $E^* \notin \mathfrak{M}(\Lambda)$ then there is a*

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finite-valued regular outer measure Λ^* *in* X *such that* $\mathfrak{M}(\Lambda)\subset\mathfrak{M}(\Lambda^*)$ *and* $E^*\notin\mathfrak{M}(\Lambda^*)$.

PROOF. Since $E^*\notin\mathfrak{M}(\Lambda)$ there is a pair of subsets P^*, Q^* of X such that $P^*\subset E^*$, $Q^*\subset X-E^*$ and

$$(1) \qquad \Lambda(P^*\cup Q^*) < \Lambda(P^*) + \Lambda(Q^*) < \infty.$$

Set $R^*=P^*\cup Q^*$ and for each subset E of X set

$$\Lambda^*(E) = \text{gr.l.b. } \Lambda\,(H\cap R^*) \text{ for } E\subset H,\ H\in\mathfrak{M}(\Lambda).$$

It follows easily that Λ^* is an outer measure in X and for $E\subset X$ there is an $H\in\mathfrak{M}(\Lambda)$ such that $E\subset H$, $\Lambda^*(E)=\Lambda(H\cap R^*)=\Lambda^*(H)$.

For $E\in\mathfrak{M}(\Lambda)$ let P, Q be a pair of subsets of X with $P\subset E$, $Q\subset X-E$. Let $H\in\mathfrak{M}(\Lambda)$ be such that $P\cup Q\subset H$ and $\Lambda^*(P\cup Q)=\Lambda(H\cap R^*)$. Since $E\in\mathfrak{M}(\Lambda)$, $P\subset H\cap E\in\mathfrak{M}(\Lambda)$, $Q\subset H\cap(X-E)\in\mathfrak{M}(\Lambda)$,

$$(2) \qquad \begin{aligned}\Lambda^*(P\cup Q) &= \Lambda(H\cap R^*) = \Lambda(H\cap E\cap R^*)\\ &\quad + \Lambda[H\cap(X-E)\cap R^*] \geqq \Lambda^*(P)+\Lambda^*(Q).\end{aligned}$$

By (iii) of §1 the equality sign holds in (2). Thus $E\in\mathfrak{M}(\Lambda^*)$ and $\mathfrak{M}(\Lambda)\subset\mathfrak{M}(\Lambda^*)$. From this fact it follows that Λ^* is regular. From (1) $\Lambda^*(X)=\Lambda(R^*)<\infty$. Since $\Lambda^*(P^*\cup Q^*)=\Lambda(P^*\cup Q^*)<\Lambda(P^*)+\Lambda(Q^*)\leqq\Lambda^*(P^*)+\Lambda^*(Q^*)$, it follows that $E^*\notin\mathfrak{M}(\Lambda^*)$.

The proof of the theorem stated in §1 is now immediate. Assume that $\mathfrak{S}[\mathfrak{M}(\Lambda)]=\mathfrak{M}(\Lambda)$ for every finite-valued regular outer measure Λ. Let Λ be an outer measure and let E^* be a set not in $\mathfrak{M}(\Lambda)$. By the preceding lemma there is a finite-valued regular outer measure Λ^* such that

$$(3) \qquad \mathfrak{M}(\Lambda)\subset\mathfrak{M}(\Lambda^*), \qquad E^*\notin\mathfrak{M}(\Lambda^*).$$

Since $\mathfrak{S}[\mathfrak{M}(\Lambda^*)]=\mathfrak{M}(\Lambda^*)$ by assumption, from (3) and condition (b) of §1, $\mathfrak{S}[\mathfrak{M}(\Lambda)]\subset\mathfrak{S}[\mathfrak{M}(\Lambda^*)]=\mathfrak{M}(\Lambda^*)$ and $E^*\notin\mathfrak{S}[\mathfrak{M}(\Lambda)]$. Since E^* was any set not in $\mathfrak{M}(\Lambda)$, $\mathfrak{S}[\mathfrak{M}(\Lambda)]\subset\mathfrak{M}(\Lambda)$. Therefore, by condition (a) of §1, $\mathfrak{S}[\mathfrak{M}(\Lambda)]=\mathfrak{M}(\Lambda)$.

BIBLIOGRAPHY

1. C. Kuratowski, *Topologie*, I, Warsaw-Lwow, 1933.
2. S. Saks, *Theory of the integral*, New York, Stechert, 1937.

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