AUTOMORPHISMS OF PRODUCTS OF MEASURE SPACES

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1. **Introduction.** Let S be the measure-theoretic product of a (not necessarily countable) family of unit intervals, $S=\prod I_\alpha$, $\alpha\in A$. In this paper we shall prove that S has the following "realization" property: every "set automorphism" ϕ of S may be induced by some "point automorphism" T of S. (For the terms used, see below.) The particular case of this theorem in which ϕ is measure-preserving shows that S has "sufficiently many measure-preserving transformations" in the terminology of Halmos and von Neumann [**1**, p. 340].

When the number of factors I_α is countable, this theorem reduces, in the measure-preserving case at least, to a known property of normal measure spaces [**2**, p. 582]. The method of proof of the general theorem will apply, more generally, to any product of measure spaces in which (a) each factor has a separating sequence, and is of total measure 1, (b) every sub-product of countably many factors has the "realization" property itself. Thus, for example, any product of 2-point factors (of measure 1) will also have the realization property. We could even allow a finite number of the factors to have infinite (but σ-finite) measure, for the transformations considered need not preserve measure. However, we shall restrict attention to the theorem as first stated.

One feature needs remark. When the number of factors is countable, T is uniquely determined by ϕ, in the sense that if T_1 and T_2 are point-automorphisms which induce the same set-automorphism ϕ, then T_1 and T_2 can differ on a null set at most. But when $S=\prod I_\alpha$, $\alpha\in A$, where A is uncountable, T is by no means unique in this sense. For example, if T_1 is the identity transformation on S and T_2 the transformation which interchanges the coordinate values 0 and 1 wherever they occur, then T_1 and T_2 both induce the identity set-automorphism. But the set on which T_1 and T_2 differ can be shown to be nonmeasurable, having outer measure 1 and inner measure 0.

2. **Notation.** Let S be any measure space, and E its algebra of measurable sets modulo null sets. Thus if X is a measurable subset of S, its class modulo null sets, denoted by $\{X\}$ or x, is a typical member of E. If S' is another measure space, with E' as its measure algebra,

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a *point isomorphism*[1] T from S to S' is a 1-1 mapping of S onto S' such that both T and T^{-1} take (i) measurable sets into measurable sets, (ii) null sets into null sets. (In the cases we are mainly concerned with, (ii) is a consequence of (i).) A *set isomorphism*[1] ϕ from S toS' is simply an isomorphism from E to E', that is, a 1-1 mapping of E onto E' which preserves suprema and complements, but not necessarily measure. Thus every T *induces* a ϕ by the rule $\phi(x)=\{T(X)\}$, $X\in x$. When $S=S'$ and $E=E'$ we speak of point and set *automorphisms*.

Throughout what follows, we assume $S=\prod I_\alpha$, $\alpha\in A$, where each I_α is a unit interval of real numbers. For each nonempty $B\subset A$, we write $S(B)$ for the partial product $\prod I_\alpha$, $\alpha\in B$, using $S(B)$ to denote both the product *set* and the measure space on it. For simplicity of notation, we also disregard the order of the factors, writing e.g. $S=S(A)=S(B)\times S(A-B)$. $\emptyset$ is used for the empty set, and we assume throughout that $A\neq\emptyset$.

If $C\subset B\subset A$, the "projection" π_{BC}: $S(B)\to S(C)$ is defined as usual by $\pi_{BC}(p)=q$, where $p\in S(B)$ and the αth coordinate q_α of q is p_α, $\alpha\in C$. When $B=A$, π_{AC} is abbreviated to π_C.

S^B denotes the family of "cylinders" on the measurable subsets of $S(B)$, i.e., of sets $\pi_B^{-1}(X)=X\times S(A-B)$ where X is a measurable subset of $S(B)$. The algebra of measurable sets modulo null sets of $S(B)$ will be written $E(B)$, and that of S^B (modulo null sets of $S(A)$) will be written E^B. It is well known (but not completely trivial) that π_B induces a measure-preserving isomorphism, which we denote by π_B also, from E^B to $E(B)$.

3. Some lemmas.

LEMMA 1. *Let E_1, E_2 be the measure algebras of two σ-finite measure spaces S_1, S_2, and let E_3 be the measure algebra of $S_1\times S_2$. Then, given automorphisms ψ_1, ψ_2 of E_1, E_2, there is a unique automorphism ψ_3 of E_3 such that*

$$\psi_3(x\times y)=\psi_1(x)\times\psi_2(y)\qquad (x\in E_1,\ y\in E_2).$$

The units e_1, e_2, of E_1, E_2, may be partitioned into disjoint elements $a_{1n}\in E_1$, $a_{2n}\in E_2$ $(n=1, 2, \cdots)$, of finite measure, such that whenever $y\leqq a_{in}$ $(i=1, 2)$ we have $(1/n)$ meas $y\leqq$ meas $\psi_i(y)\leqq n$ meas y. It is a routine matter to extend the correspondence

$$\psi_3(x\times y)=\psi_1(x)\times\psi_2(y)\qquad (x\leqq a_{1m},\ y\leqq a_{2n})$$

[1] This terminology differs from that in [1], where isomorphisms are required to be measure-preserving.

to an isomorphism between the ideals A_{mn} and B_{mn} of E_3, where A_{mn} consists of all elements $\leq a_{1m} \times a_{2n}$ and B_{mn} of all elements $\leq \psi_1(a_{1m}) \times \psi_2(a_{2n})$, and thence to extend ψ_3 to all of E_3. The proof that ψ_3 has the stated property, and of its uniqueness, presents no difficulty.

LEMMA 2. *Let ϕ be a set automorphism of $S = \prod I_\alpha$, and let T be a 1-1 mapping of S onto itself such that, for each finite set C of suffixes α, and for each measurable set K of S^C, $T(K) \in \phi\{K\}$ and $T^{-1}(K) \in \phi^{-1}\{K\}$. Then T is a point automorphism of S, and induces ϕ.*

Let $\mathcal{B}$ be the Borel field generated by all sets of the form K, i.e., by all cylinder sets which are based on measurable sets in *finite* products of I_α's. We recall that $\mathcal{B}$ generates S in the following sense: (1) each measurable subset of S differs from some set in $\mathcal{B}$ by a null set, (2) each null subset of S is contained in some null set in $\mathcal{B}$. Now it is easy to see that the measurable subsets X of S which have the property:

$$T(X) \in \phi\{X\} \quad \text{and} \quad T^{-1}(X) \in \phi^{-1}\{X\}, \tag{3}$$

form a Borel field. Hence every set in $\mathcal{B}$ has this property. From (2) it follows that $T(X)$ and $T^{-1}(X)$ are null whenever X is null, and hence (1) shows that every measurable X has the property. In particular, $T(X)$ and $T^{-1}(X)$ are measurable if X is, so T is a point automorphism of S; and clearly T induces ϕ.

DEFINITION. Let ϕ be a given automorphism of E. A set $B \subset A$ will be called "*invariant*" (under ϕ) if $\phi(E^B) = E^B$. Restricted to E^B, ϕ will then be an automorphism of E^B.

LEMMA 3. *Each countable set $B \subset A$ is contained in some countable subset $\overline{B}$ of A which is invariant.*[2]

Let $B_0 = B$, and take a countable basis b_{0m}, $m = 1, 2, \cdots$, for E^{B_0} (apply π_B^{-1} to a countable basis for $E(B)$). Consider the elements $\phi^n(b_{0m})$ ($n = 0, \pm 1, \pm 2, \cdots$). The properties (1) and (2) stated at the beginning of the proof of Lemma 2, show that each of these measure classes contains a set which is a cylinder on only countably many coordinates; hence there is a countable set $B_1 \subset A$ such that every $\phi^n(b_{0m})$ is in E^{B_1}. Take a countable basis b_{1m}, $m = 1, 2, \cdots$, for E^{B_1}, and repeat the process, obtaining a countable set B_2; and so on. Then $\bigcup B_k$ ($k = 0, 1, \cdots$) is the countable invariant set required.

LEMMA 4. *Every set-automorphism of the unit interval can be induced by a point-automorphism.*

[2] More generally, each $B \subset A$ is contained in an invariant set $\overline{B} \subset A$ of cardinal $\leq \max(\aleph_0, |B|)$. It can be shown that, given B ($\neq \emptyset$), there is a *smallest* $\overline{B}$.

Let ϕ be a set-automorphism of the unit interval I, and for each $t\in I$ let I_t denote the interval from 0 to t. The mapping U defined by $U(t) = \text{meas}\,\phi\{I_t\}$ is a point-automorphism of I, and induces a set-automorphism ψ. Since U maps I_t onto the interval from 0 to $U(t)$, we have $\text{meas}\,\psi\{I_t\} = \text{meas}\,\phi\{I_t\}$ for each $t\in I$; it follows that $\psi(x)$ and $\phi(x)$ have the same measure for each class x in the measure algebra of I. Thus $\psi^{-1}\phi$ is a *measure-preserving* set-automorphism of I, and [2] there is a point-automorphism V of I which induces $\psi^{-1}\phi$. Then UV is a point-automorphism of I which induces ϕ.

LEMMA 5. *Let $S = \prod I_\alpha$, $\alpha\in A$, and let B be any subset of A for which $A-B$ is countable. Suppose ϕ is a set-automorphism of S which, restricted to E^B, is the identity mapping of E^B. Then there exists a point-automorphism T of S which induces ϕ, and which satisfies $\pi_B T = \pi_B$.*

Note that the hypothesis on ϕ implies that B is invariant.

Write $C = A-B$; by Lemma 3, $C\subset\overline{C}\subset A$ where $\overline{C}$ is countable and invariant. Writing $D = \overline{C} - C$, we have $D\subset B$. Now, since $\phi(E^{\overline{C}}) = E^{\overline{C}}$, ϕ induces an automorphism $\phi_1 = \pi_{\overline{C}}\phi\pi_{\overline{C}}^{-1}$ on $E(\overline{C})$. And, because $\overline{C}$ is countable, $S(\overline{C})$ is isomorphic, under a measure-preserving point-isomorphism, to I. Hence, by Lemma 4, there exists a point-automorphism T_1 of $S(\overline{C})$ which induces ϕ_1.

Suppose first that $D\neq\emptyset$. Then we can regard $S(\overline{C})$ as $S(D)\times S(C)$, and have (because ϕ is the identity on E^B)

$$\phi(e(A-\overline{C})\times x\times e(C)) = [e(A-\overline{C})\times x]\times e(C) \quad \text{for each } x\in E(D),$$

$e(A-\overline{C})$ and $e(C)$ denoting the unit elements of $E(A-\overline{C})$, $E(C)$. Hence $\phi_1(x\times e(C)) = \pi_{\overline{C}}\phi\pi_{\overline{C}}^{-1}(x\times e(C)) = x\times e(C)$ for each $x\in E(D)$. Thus, for each measurable subset X of $S(D)$, $T_1(X\times S(C))$ differs from $X\times S(C)$ by a null set. Apply this to the sets $T_1^i(X_n)$ $(i=0, \pm 1, \cdots, n=1, 2, \cdots)$ in turn, where $X_1, X_2, \cdots,$ forms a separating sequence of measurable sets in $S(D)$; we obtain countably many null sets with union N, say. Then N is null, $T_1(N) = N = T_1^{-1}(N)$, and

$$T_1[(X_n\times S(C)) - N] = (X_n\times S(C)) - N \qquad (n = 1, 2, \cdots).$$

Define a transformation T_2 on $S(\overline{C})$ by:

$$T_2(p) = T_1(p) \text{ if } p\in S(\overline{C}) - N; \qquad T_2(p) = p \text{ if } p\in N.$$

Clearly T_2 is another point-automorphism of $S(\overline{C})$ which also induces ϕ_1; and we now have

$$T_2(X_n\times S(C)) = X_n\times S(C) \qquad (n = 1, 2, \cdots).$$

As $X_1, X_2, \cdots,$ is a separating sequence, it follows that

$$T_2(p \times S(C)) = p \times S(C) \qquad (p \in S(D)),$$

and hence that

$$T_2(X \times S(C)) = X \times S(C) \quad \text{for every } X \subset S(D).$$

Finally we define T by

$$T(p \times q) = p \times T_2(q) \quad \text{where} \quad p \in S(A - \bar{C}) \quad \text{and} \quad q \in \bar{C}.$$

Then T is clearly a point-automorphism of S, and it is easily seen that $\pi_B T = \pi_B$. To show that T induces ϕ, it is enough to prove (by a double application of Lemma 1) that if $X \in x \in E(A - \bar{C})$, $Y \in y \in E(D)$, $Z \in z \in E(C)$, then $T(X \times Y \times Z) \in \phi(x \times y \times z)$, and this can be done by a straightforward calculation.

If $D = \varnothing$, we define $T_2 = T_1$ on $S(\bar{C}) = S(C)$, and define T as before; only minor (simplifying) adjustments are needed in the argument.

LEMMA 6. *Let $S = \prod I_\alpha$, $\alpha \in A$, and let B be any subset of A for which $A - B$ is countable. Suppose ϕ is a set-automorphism of S, and that B is invariant under ϕ, so that ϕ restricted to E^B induces an automorphism ϕ' of $E(B)$. Then, given any point-automorphism T' of $S(B)$ which induces ϕ', there exists a point-automorphism T of S which induces ϕ, and which satisfies $\pi_B T = T' \pi_B$.*

Lemma 5 is the special case when $T' =$ identity, and we reduce the general case to the special one. By Lemma 1, there exists an automorphism ψ of E such that

$$\psi(x \times y) = \phi'(x) \times y, \qquad x \in E(B),\ y \in E(A - B).$$

Then $\theta = \phi\psi^{-1}$ is also an automorphism of E; and, using the fact that $\phi' = \pi_B \phi \pi_B^{-1}$, it is easy to see that θ is the identity mapping on E^B. By Lemma 5, there exists a point-automorphism T^* of S which induces θ on E, and which satisfies $\pi_B T^* = \pi_B$. Define

$$T(p \times q) = T^*(T'(p) \times q), \qquad p \in S(B),\ q \in S(A - B).$$

Then T is clearly a point-automorphism of S, and it is a straightforward matter to verify that T has the desired properties.

4. THEOREM. *Let $S = \prod I_\alpha$, $\alpha \in A$, and let ϕ be a set-automorphism of S. Then there exists a point-automorphism T of S which induces ϕ.*

Consider the family of ordered pairs (B_λ, T_λ) where (i) B_λ is a subset of A which is invariant under ϕ, (ii) T_λ is a point-automorphism of $S(B_\lambda)$, and (iii) the automorphisms of $E(B_\lambda)$ induced by ϕ restricted to E^{B_λ} (i.e., $\pi_{B_\lambda} \phi \pi_{B_\lambda}^{-1}$), and by T_λ, are the same. Say that $(B_\lambda, T_\lambda) < (B_\mu, T_\mu)$ provided that $B_\lambda \subset B_\mu$ and $\pi_{\mu\lambda} T_\mu = T_\lambda \pi_{\mu\lambda}$ on $S(B_\mu)$.

Here $\pi_{\mu\lambda}$ is used as an abbreviation for $\pi_{B_\mu B_\lambda}$; similarly we shall abbreviate π_{B_λ} to π_λ. The partial ordering so defined is clearly transitive. Further, every linearly ordered subfamily $\{(B_\mu, T_\mu), \mu \in M\}$ has an upper bound in the family. To see this, define $B' = \bigcup B_\mu$; this is an invariant subset of A (under ϕ). Given $p \in S(B')$ and $\alpha \in B'$, pick any $B_\mu \ni \alpha$, and let q_α be the αth coordinate of $T_\mu(\pi_\mu(p))$. It is easy to see that q_α is independent of the choice of μ, and we define $T'(p)$ to be the point of $S(B')$ having αth coordinate q_α ($\alpha \in B'$). A straightforward calculation, using Lemma 2, shows that (B', T') is a member of our family, and that $(B_\mu, T_\mu) < (B', T')$ for each $\mu \in M$.

By Zorn's lemma, it follows that there is a maximal member (B, T) of the family (note that the family is not vacuous, from Lemmas 3 and 4). It is enough to prove that $B = A$, for condition (iii) above then shows that T induces ϕ. Suppose not, and pick $\alpha \in A - B$; by Lemma 3 there is a countable set $D \subset A$, invariant under ϕ, which contains α. Let $B^* = B \cup D$; then B^* is also invariant, and $\phi^* = \pi_{B^*} \phi \pi B^{*-1}$ is an automorphism of $S^* = S(B^*)$. We apply Lemma 6 to the product space S^*, with invariant subset B, set-automorphism ϕ^* and point-automorphism T, obtaining a point-automorphism T^* of S^* which induces ϕ^* and satisfies $\pi_{B^*B} T^* = T \pi_{B^*B}$. But now (B^*, T^*) is a member of the family defined above, and it contradicts the maximality of (B, T), since $(B, T) < (B^*, T^*)$ and $B \neq B^*$. This contradiction establishes the theorem.

References

1. P. Halmos and J. von Neumann, *Operator methods in classical mechanics* II, Ann. of Math. (2) vol. 43 (1942) pp. 332–350.

2. J. von Neumann, *Einige Sätze über messbare Abbildungen*, ibid. vol. 33 (1932) pp. 574–586.

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