TOTALLY NONCONNECTED IM KLEINEN CONTINUA

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If a connected topological space T' is not connected im kleinen at the point p of T', then there is some open set D' containing p such that p is a boundary point of the p-component of D'.

This paper shows that much stronger conditions than those above hold at some points of a Baire topological continuum[1] which is not connected im kleinen at any point of a certain domain intersection subset. Limits to results of this kind are shown by an example of a bounded plane continuum that is connected im kleinen at each point of a dense inner limiting (i.e., G_δ) subset but is not locally connected at any point.

DEFINITION. A topological continuum T' is *totally nonconnected im kleinen* on a subset A of T' if T' is not connected im kleinen at any point of A (i.e., if each point p of A is contained in some open subset U of T' such that p is a boundary point, relative to T', of the p-component of U).

DEFINITION. If (1) T' is a topological continuum, (2) Z' is the least cardinal number of any topological basis for T', (3) the set U is an open subset of T', and (4) the subset A of U is the nonvacuous common part of not more than Z' open subsets of T', each dense in U, then A is a *dense-domain intersection subset* of U (relative to T'). If, in addition to (1), (2) and (3), U is not the sum of Z', or fewer than Z', closed nowhere dense subsets of T', then T' is *Baire topological* on U. If T' is Baire topological on each open subset of U, then T' is *locally Baire topological* on U.

THEOREM 1. *If the topological continuum T' is Baire topological on the open subset U, then U contains an open subset V such that T' is locally Baire topological on V.*

PROOF. Let $R_1', R_2', \cdots, R_i', \cdots$ be a most economical well-ordering of a basis for T' of minimum cardinality. Assume T' is not locally Baire topological on any open subset of U. Then there is a subsequence $R'_{N(1)}, R'_{N(2)}, \cdots, R'_{N(i)}, \cdots$ of $R_1', R_2', \cdots, R_i', \cdots$ such that, for each i, there is a dense-domain intersection subset I_i of T' such that $I_i \cdot R'_{N(i)} = 0$ and each open subset of U contains some term of $R'_{N(1)}, R'_{N(2)}, \cdots, R'_{N(i)}, \cdots$. But $\prod_i I_i$ is the common part

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[1] The terminology of this paper is that of [1].

of not more than Z' dense-domain intersection subsets of T, where Z' is the cardinality of $R_1', R_2', \cdots, R_i', \cdots$. Consequently [1, Theorem 3], the set $\prod_i I_i$ is a dense-domain intersection subset of T. But [1, Theorem 1], the closure of any dense-domain intersection subset of T contains an open subset of U, and therefore, the closure of $\prod_i I_i$ contains an open subset of U. This is a contradiction.

Standing notation. T is a topological continuum which is locally Baire topological on the open set D and is totally nonconnected im kleinen on a dense-domain intersection subset I of D. The least cardinal number of a basis for T is Z. The collection R of regions is a basis for T of cardinality Z. The sequence $R_1, R_2, \cdots, R_i, \cdots$, is a most economical well-ordering of R.

LEMMA 1. *Let U be an open subset of D and let S be the set of all points p such that p is in $\overline{U}-U$ or p is in U and p is a boundary point of the p-component of U. Then S is closed.*

PROOF. Assume there is a point q in $\overline{S}$ which is not in S. Then q is in U since $\overline{U}\supset\overline{S}$ and each point of $\overline{U}-U$ is in S. Therefore, each region containing q contains a point r of $S\cdot U$ and hence contains a point of U not in the q-component of U. Consequently, q is in S. This is a contradiction and so S is closed.

LEMMA 2. *Let V be an open subset of D. Then V contains an open set no component of which contains an open set.*

PROOF. Let $R_{M(1)}, R_{M(2)}, \cdots, R_{M(i)}, \cdots$ be the subsequence of $R_1, R_2, \cdots, R_i, \cdots$ consisting of those terms contained in V. For each i, let S_i be the set related to $R_{M(i)}$ as S is related to U in Lemma 1. The continuum T is totally nonconnected im kleinen on the dense-domain intersection subset $I\cdot V$ of V and therefore $\sum_i S_i\supset I\cdot V$. But, since I is a dense-domain intersection subset of D, the set $D-I$ is contained in the sum of not more than Z closed sets, each nowhere dense in D. Hence, if each term of $S_1, S_2, \cdots$ were nowhere dense, then V would be contained in the sum of not more than Z closed, nowhere dense subsets of T. But that can not be, since T is Baire topological on V. Consequently, for some i, the set S_i contains an open set V'. By definition of S_i, no component of the open set $V\cdot V'$ can contain an open set.

THEOREM 2. *The open set D contains an open set D', dense in D, no component of which contains an open set.*

PROOF. Let D_1 be an open subset of D, no component of which contains an open set. If i is an ordinal greater than 1 such that the closure of $\sum_{j<i} D_j$ does not contain D, then let D_i be an open subset of the complement, in D, of the closure of $\sum_{j<i} D_j$, no component of which contains an open set. Let $D' = \sum_j D_j$. Then D' is an open subset of D, dense in D, no component of which contains an open set.

THEOREM 3. *Let the topological continuum T' be locally Baire topological on the open set V. Then T' is totally nonconnected im kleinen on a dense-domain intersection subset of V if and only if there is an open subset of V, dense in V, no component of which contains an open set.*

THEOREM 4. *If T is regular then T contains a dense-domain intersection subset J of D, contained in I, such that if p is a point of J and T is nonaposyndetic at p with respect to $T-\overline{R}_i$, where R_i contains p, then the p-component of $\overline{R}_i$ does not contain an open set.*

PROOF. The collection G of complements of closures of regions of R has the Z-domain property and T is totally nonaposyndetic on I with respect to G. Let P be a dense subset of T of cardinality not greater than Z. The author has proved [1, Theorem 8] that under these circumstances there is a dense-domain intersection subset J of D, contained in I, such that if q is a point in J and T is nonaposyndetic at q with respect to $T-\overline{R}_i$ of G then $T-\overline{R}_i$ cuts (weakly) q from each point of $P\cdot\overline{R}_i$. In which case, the p-component of $\overline{R}_i$ does not contain an open set.

The following example of a plane continuum, each connected open subset of which is dense in it, shows that Theorem 3 does not hold if "totally nonconnected im kleinen on a dense-domain intersection subset of V" is replaced by "not locally connected at any point of V"; even in the case where $V=T'$.

Let $H_1, H_2, \cdots$ be the sequence of closed plane point sets defined by induction as follows. Let S denote the unit disk and let H_1 be the logical sum of the closed (topological) disk sequence $J_1, J_2, \cdots$ indicated in Figure 1. The sum of the straight portions of $J_1, \cdots, J_4$ is the limiting set of $J_1, J_2, \cdots$.

Assume H_i to be defined and let $J_1', J_2', \cdots$ be a counting of the closed disks maximal in H_i. For each natural number j, there exist two points r and s of J_j' such that $r+s$ cuts J_j' from H_i-J_j' in H_i and such that if a single point t does the same cutting then $t=r$. Let f_j be a homeomorphism of S onto J_j' such that $f_j(p)=r$ and $f_j(q)=s$. Let $H_{i+1}=\sum_j f_j(H_1)$. The homeomorphisms of S onto the closed maximal disks of the sets $H_2, H_3, \cdots$ are chosen in such a way that the diameters of disks in $H_1, H_2, \cdots$ are eventually as small as one wishes.

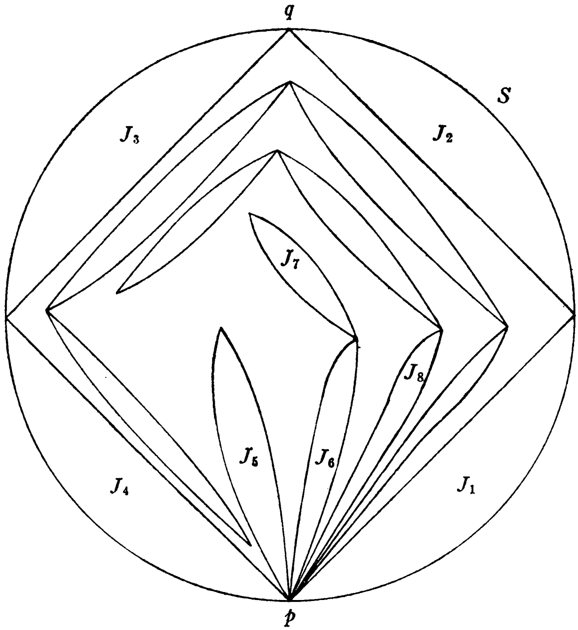

FIG. 1

Let $H = \prod_i H_i$. It is clear that H is not locally connected at any point but H is connected im kleinen at each point of the dense inner limiting subset of H consisting of the points that, for each i, are interior points (in the plane) of H_i. Hence H does not contain any open set no component of which contains an open set.

BIBLIOGRAPHY

1. E. E. Grace, *The existence of cut sets in totally nonaposyndetic continua*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 98–104.

2. J. L. Kelley, *General topology*, New York, 1955.

3. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications, vol. 13, 1932.

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