AN INVARIANT PROPERTY OF CESARI'S SURFACE INTEGRAL[1]

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1. **Introduction.** Let $A \subset E_2$ be any admissible set. (See [2] for all definitions and theorems stated here without proof.) Let T be a continuous mapping of A into E_3, $p = T(w)$, $p = (x, y, z) \in E_3$, $w = (u, v) \in E_2$. We will call $S = (T, A)$ a c.BV surface or mapping (continuous and of bounded variation) if the area $V(T, A)$ of the mapping is finite. Let τ_1, τ_2, τ_3 be the orthogonal projections of E_3 onto the oriented (y, z) plane E_{21}, (z, x) plane E_{22}, (x, y) plane E_{23}, respectively. If (T, A) is a c.BV mapping, then the plane mappings $(T_r, A) = (\tau_r T, A)$ have variations (or areas) $V(T_r, A)$ which are all finite. Each of these variations is equal to the sum of a positive and a negative variation, $V(T_r, A) = V^+(T_r, A) + V^-(T_r, A)$. A relative variation is also defined by $\mathcal{V}(T_r, A) = V^+(T_r, A) - V^-(T_r, A)$.

In this paper, given a c.BV surface (T, A), the symbol $\mathfrak{S}$ will always stand for a finite set of nonoverlapping simple polygonal regions $\pi \subset A$. Also $\sum_\pi$ will denote a sum over all polygons $\pi \in \mathfrak{S}$; $\sum_r$ will stand for a sum over $r = 1, 2, 3$. For any point or vector $d = (d_1, d_2, d_3) \in E_3$ we use $\|d\| = (d_1^2 + d_2^2 + d_3^2)^{1/2}$.

Now let $\mathfrak{S}$ be any set of polygons π for a c.BV mapping (T, A). Let π^* denote the oriented boundary of π. Then T_r maps π^* into a oriented closed curve $C_{\pi r}$ in E_{2r}. For any point $p \in E_{2r}$, let $O(p; C_{\pi r})$ be the topological index of p with respect to $C_{\pi r}$. Then $O(p; C_{\pi r})$ is Borel measurable and integrable over E_{2r}. Define $O^+ = (|O| + O)/2$, $O^- = (|O| - O)/2$, and

$$v^+(\pi, T_r) = (E_{2r}) \int O^+(p; C_{\pi r}), v^-(\pi, T_r) = (E_{2r}) \int O^-(p; C_{\pi r}),$$

$$u(\pi, T_r) = v^+ - v^-, v(\pi, T_r) = v^+ + v^-, d_\pi = (u(\pi, T_1), u(\pi, T_2), u(\pi, T_3)).$$

Now for every set $\mathfrak{S}$ we define three nonnegative indices d, m, μ, as in [2], by $d = \max\{\text{diam } T(\pi) : \pi \in \mathfrak{S}\}$; $m = \max\{ | T_r(\sum_\pi \pi^*) | : r = 1, 2, 3\}$, where the absolute value sign denotes two dimensional

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Lebesgue measure; $\mu=\max\ \{V(T, A)-\sum_\pi \|d_\pi\|,\ V(T_r, A)-\sum_\pi|u(\pi, T_r)| : r=1, 2, 3\}$.

Now let $f(p, d)=f(x, y, z, d_1, d_2, d_3)$ be a continuous function of (p, d), p in some set $K\subset E_3$ and d any point of E_3. We will call $f(p, d)$ a parametric integrand if $f(p, d)$ is positively homogeneous in d; i.e., $f(p, \alpha d)=\alpha f(p, d)$ for all $p\in K$, $\alpha\geqq 0$, $d\in E_3$.

In [1] and [2, Appendix B], L. Cesari defined his surface integral and proved its existence as in the following theorem:

THEOREM 1. *Let (T, A) be a c.BV surface. Let $f(p, d)$ be a parametric integrand defined on $K\times E_3$ such that $T(A)\subset K$ and $f(p, d)$ is bounded and uniformly continuous on $R=\{(p, d): p\in T(A), \|d\|=1\}$. Then the limit $I(T, A; f)=\lim \sum_\pi f(p_\pi, d_\pi)$ exists, where p_π is any point of $T(\pi)$, where π is an element of a set $\mathfrak{S}$, and the limit is taken as the indices d, m, μ of sets $\mathfrak{S}$ tend to zero.*

Let $\mathcal{V}_\pi=(\mathcal{V}(T_1, \pi), \mathcal{V}(T_2, \pi), \mathcal{V}(T_3, \pi))$ for any polygon of a set $\mathfrak{S}$. We will also use $\mathcal{V}(T, A)=(\mathcal{V}(T_1, A), \mathcal{V}(T_2, A), \mathcal{V}(T_3, A))$. The purpose of the present paper is to prove the following three theorems:

THEOREM 2. *Under the same hypotheses as in Theorem 1, $\lim \sum_\pi f(p_\pi, \mathcal{V}_\pi)$ exists, the limit being taken as in Theorem 1, and this limit equals $I(T, A; f)$.*

THEOREM 3. *Let (T, A) be any c.BV mapping. Let (α_{ij}) be the matrix of a linear orthogonal transformation α of E_3 onto itself. Let (T', A) be the mapping defined by $T'=\alpha T$ and let $T_r'=\tau_r T'$. Then $\mathcal{V}(T', A)=\alpha\mathcal{V}(T, A)$.*

THEOREM 4. *Let (T, A) be a c.BV surface. Let $f(p, d)$ be a parametric integrand defined on $K\times E_3$ with $T(A)\subset K$. Let $f(p, d)$ be bounded and uniformly continuous on $R=\{(p, d): p\in T(A), \|d\|=1\}$. Let α be a linear orthogonal transformation of E_3 onto itself. Let $g(p,d)=f(\alpha^{-1}p, \alpha^{-1}d)$ on $(\alpha K)\times E_3$ and let $(T', A)=(\alpha T, A)$. Then $I(T', A; g)$ exists and equals $I(T, A; f)$.*

The integral $I(T, A; f)$ has been used in the calculus of variations by A. G. Sigalov, J. M. Danskin, J. Cecconi, and V. E. Bononcini.

2. **The proof of Theorem 2.** We first prove the following lemma.

LEMMA. *Let (T, A) be any c.BV surface. Let $\mathfrak{S}$ be any set of polygons $\pi\subset A$ with index μ. Then $\sum_\pi |\mathcal{V}(T_r, \pi)-u(\pi, T_r)|\leqq\mu$, $r=1, 2, 3$ and $\sum_\pi|\|\mathcal{V}_\pi\|-\|d_\pi\||\leqq\sum_\pi\|\mathcal{V}_\pi-d_\pi\|\leqq 3\mu$.*

PROOF.

$$\sum_\pi |\mathfrak{V}(T_r,\pi)-u(\pi,T_r)|$$
$$=\sum_\pi |V^+(T_r,\pi)-V^-(T_r,\pi)-v^+(\pi,T_r)+v^-(\pi,T_r)|$$
$$\leqq \sum_\pi |V^+(T_r,\pi)-v^+(\pi,T_r)| + \sum_\pi |V^-(T_r,\pi)-v^-(\pi,T_r)|$$
$$=\sum_\pi [V^+(T_r,\pi)-v^+(\pi,T_r)] + \sum_\pi [V^-(T_r,\pi)=v^-(\pi,T_r)]$$
$$=\sum_\pi [V(T_r,\pi)-v(\pi,T_r)] \leqq V(T_r,A)-\sum_\pi |u(\pi,T_r)| \leqq \mu,$$

where we have used the overadditivity of the area function V. Also $\sum_\pi | \|\mathfrak{V}_\pi\| - \|d_\pi\| | \leqq \sum_\pi \|\mathfrak{V}_\pi - d_\pi\| \leqq \sum_\pi \sum_r |\mathfrak{V}(T_r,\pi)-u(\pi,T_r)| \leqq 3\mu$.

PROOF OF THEOREM 2. Let M be a constant,

$$M \geqq \sup\{|f(p,d)| : (p,d) \in R\}, \qquad M \geqq V(T,A).$$

Given $\epsilon>0$, let $\rho>0$ be such that $\rho<1$ and $|f(p,d)-f(p,d^*)|<\epsilon/17M$ when (p,d), $(p,d^*)\in R$, $\|d-d^*\|<\rho$. Let $\sigma>0$ be such that $|I(T,A;f)-\sum_\pi f(p_\pi,d_\pi)|<\epsilon/17$ when $\mathfrak{S}$ has indices d, m, $\mu<\sigma$. Let $\lambda=\min(\sigma,\rho,\epsilon\rho/17M)$ and let $\mathfrak{S}$ be any set of polygons with indices d, m, $\mu<\lambda$. Let $\sum'$ denote a sum over all $\pi\in\mathfrak{S}$ for which

$$\| d_\pi/\|d_\pi\| - \mathfrak{V}_\pi/\|\mathfrak{V}_\pi\| \| < \rho,$$

$\sum''$ denote a sum over all other $\pi\in\mathfrak{S}$. (Replace any $d_\pi/\|d_\pi\|$ by zero if $\|d_\pi\|=0$. Similarly for $\mathfrak{V}_\pi$.) Then for p_π any point of $T(\pi)$,

$$\sum_\pi [f(p_\pi,d_\pi)-f(p_\pi,\mathfrak{V}_\pi)] = \sum_\pi f(p_\pi,d_\pi)/\|d_\pi\|)[\|d_\pi\|-\|\mathfrak{V}_\pi\|]$$
$$+\left(\sum{}'+\sum{}''\right)[f(p_\pi,d_\pi/\|d_\pi\|)-f(p_\pi,\mathfrak{V}_\pi/\|\mathfrak{V}_\pi\|)]\|\mathfrak{V}_\pi\|$$
$$= s_1+s_2+s_3.$$

From the lemma, $|s_1|<3M\lambda\leqq 3\epsilon/17$. Also

$$|s_2| \leqq \sum{}' \{ |f(p_\pi,d_\pi/\|d_\pi\|)-f(p_\pi,\mathfrak{V}_\pi/\|\mathfrak{V}_\pi\|)| \, \|\mathfrak{V}_\pi\| \}$$
$$\leqq (\epsilon/17M)\sum{}' \|\mathfrak{V}_\pi\| \leqq \epsilon/17.$$

We have used the inequality $\|\mathfrak{V}_\pi\| \leqq V(T,\pi)$ which follows from

$$\|\mathfrak{V}_\pi\| = \left\{\sum_r [V^+(T_r,\pi)-V^-(T_r,\pi)]^2\right\}^{1/2} \leqq \left\{\sum_r V^2(T_r,\pi)\right\}^{1/2}$$
$$\leqq V(T,\pi),$$

the last inequality being a fundamental inequality of surface area theory, [2, §18.10]. Using the lemma again,

$$\rho \sum\nolimits'' \|\mathcal{V}_\pi\| \leqq \sum\nolimits'' \|\mathcal{V}_\pi\| \, \big\| d_\pi/\|d_\pi\| - \mathcal{V}_\pi/\|\mathcal{V}_\pi\| \big\| \leqq \sum_\pi \big\| \mathcal{V}_\pi - d_\pi \|\mathcal{V}_\pi\|/\|d_\pi\| \big\|$$

$$\leqq \sum_\pi \{ \|\mathcal{V}_\pi - d_\pi\| + | 1 - \|\mathcal{V}_\pi\|/\|d_\pi\| | \, \|d_\pi\| \}$$

$$\leqq \sum_\pi \{ \|\mathcal{V}_\pi - d_\pi\| + | \|d_\pi\| - \|\mathcal{V}_\pi\| | \} < 6\lambda \leqq 6\epsilon\rho/17M.$$

Thus $\sum'' \|\mathcal{V}_\pi\| < 6\epsilon/17M$ and $|s_3| \leqq 2M \sum'' \|\mathcal{V}_\pi\| < 12\epsilon/17$. Therefore

$$\Big| I(T, A; f) - \sum_\pi f(p_\pi, \mathcal{V}_\pi) \Big|$$

$$\leqq \Big| I(T, A; f) - \sum_\pi f(p, d_\pi) \Big| + \Big| \sum_\pi [f(p_\pi, d_\pi) - f(p_\pi, \mathcal{V}_\pi)] \Big|$$

$$< \epsilon/17 + |s_1| + |s_2| + |s_3| < \epsilon/17 + 3\epsilon/17 + \epsilon/17 + 12\epsilon/17 = \epsilon.$$

Thus $I(T, A; f) = \lim \sum_\pi f(p_\pi, \mathcal{V}_\pi)$ as stated.

Thus Cesari's integral may be defined as a limit of sums of the form given in Theorem 2. G. M. Ewing in [3] has defined a surface integral using essentially the sums of Theorem 2 for a particular class of mappings. However, every (Fréchet) surface of finite area has a representation of the type needed by Ewing by a theorem of Cesari [2, p. 544]. (See [2, Chapter IX] for the definition of Fréchet surface and a discussion of representation theory.)

3. The invariance of the integral under rotations. The proof of Theorem 4 is a simple application of Theorem 3, which says that the coordinates of the relative variation vector $\mathcal{V}(T, A)$ transform covariantly to the coordinates of points in E_3. The vectors d_π used in Theorem 1 do not have this property in general and the definition of $I(T, A; f)$ in Theorem 1 is not as suited to the proof of Theorem 4 as the expression for $I(T, A; f)$ given in Theorem 2.

PROOF OF THEOREM 3. Suppose first that the domain A is a simple closed Jordan region and T maps A^* onto a rectifiable curve with representation $x = x(t)$, $y = y(t)$, $z = z(t)$, $0 \leqq t \leqq 1$. Then from [2; p. 104, p. 203], we have $\mathcal{V}(T_1, A) = (E_{21}) \int n(p; T_1, A) = (E_{21}) \int O(p; C_1) = 1/2 \oint_{C_1} (y dz - z dy)$ where $O(p, C_1)$ is the topological index of any point $p \in E_{21}$ with respect to the oriented closed curve $C_1 = T_1(A^*)$ and $n(p; T_1, A)$ is the relative multiplicity function of (T_1, A). Similarly for $\mathcal{V}(T_2, A)$, $\mathcal{V}(T_3, A)$.

Now suppose A is a finitely connected closed Jordan region of connectivity ν, $A = J_0 - (J_1^0 + J_2^0 + \cdots + J_\nu^0)$, and T maps each boundary curve J_λ^*, $\lambda = 0, 1, 2, \cdots, \nu$, onto a rectifiable curve C_λ with representation $x_\lambda(t)$, $y_\lambda(t)$, $z_\lambda(t)$, $0 \leqq t \leqq 1$. Then from the discussion above for simple regions and from [2, §12.15] it follows that $\mathfrak{V}(T_1, A) = 1/2 \sum_\lambda \oint (y_\lambda dz_\lambda - z_\lambda dy_\lambda)$ and similarly for $\mathfrak{V}(T_2, A)$, $\mathfrak{V}(T_3, A)$. Now $\alpha T = T'$ maps J_λ^* onto an oriented rectifiable curve C_λ' and a representation of C_λ' is $\xi_\lambda(t) = \alpha_{11}x_\lambda(t) + \alpha_{12}y_\lambda(t) + \alpha_{13}z_\lambda(t)$, $\eta_\lambda(t) = \alpha_{21}x_\lambda(t) + \alpha_{22}y_\lambda(t) + \alpha_{23}z_\lambda(t)$, $\zeta_\lambda(t) = \alpha_{31}x_\lambda(t) + \alpha_{32}y_\lambda(t) + \alpha_{33}z_\lambda(t)$. As above, if $T_r' = \tau_r T'$ then $\mathfrak{V}(T_1', A) = 1/2 \sum_\lambda \oint (\eta_\lambda d\zeta_\lambda - \zeta_\lambda d\eta_\lambda)$. A simple calculation shows that $\oint (\eta_\lambda d\zeta_\lambda - \zeta_\lambda d\eta_\lambda) = \alpha_{11} \oint (y_\lambda dz_\lambda - z_\lambda dy_\lambda) + \alpha_{12} \oint (z_\lambda dx_\lambda - x_\lambda dz_\lambda) + \alpha_{13} \oint (x_\lambda dy_\lambda - y_\lambda dx_\lambda)$. Thus $\mathfrak{V}(T_1', A) = \alpha_{11}\mathfrak{V}(T_1, A) + \alpha_{12}\mathfrak{V}(T_2, A) + \alpha_{13}\mathfrak{V}(T_3, A)$. Similarly for $\mathfrak{V}(T_2', A)$, $\mathfrak{V}(T_3', A)$ and these relations may be combined as $\mathfrak{V}(T', A) = \alpha\mathfrak{V}(T, A)$.

From [2, §12.14], V^+ and V^- are additive on a finite union of disjoint closed admissible sets. Thus if $A = \sum R$ where the sets R are disjoint closed Jordan regions and (T, A) is a c.BV mapping such that T maps every boundary curve of the regions R into a rectifiable curve, then $\mathfrak{V}(T_r, A) = \sum_R \mathfrak{V}(T_r, R)$, $\mathfrak{V}(T_r', A) = \sum_R \mathfrak{V}(T_r', A)$, and $\mathfrak{V}(T', A) = \alpha\mathfrak{V}(T, A)$ since $\mathfrak{V}(T', R) = \alpha\mathfrak{V}(T, R)$ for every R by the discussion above.

Now let (T, A) be a c.BV mapping from an arbitrary admissible set A. We will define, as in [2], a figure F as a finite union of disjoint finitely connected polygonal regions. A continuous surface (T, F) is quasilinear if F can be decomposed into a finite set of nonoverlapping polygonal regions such that T is linear but not necessarily homogeneous on each of them. Also we say that a sequence of figures $\{F_n\}$ invades A if $F_n \subset A$, $F_n^0 \subset F_{n+1}^0$ for all n, and $\lim F_n^0 = A^0$. Now from [2, p. 37] there is a sequence $\{(T_n, F_n)\}$ of quasilinear mappings such that $\{F_n\}$ invades A, $\delta_n = \sup\{\|T_n(w) - T(w)\| : w \in F_n\}$ converges to zero, and $V(T_n, F_n)$ converges to $V(T, A)$. As a consequence, $V(T_{nr}, F_n)$ converges to $V(T_r, A)$, [2, p. 393], and finally $\mathfrak{V}(T_{nr}, F_n)$ converges to $\mathfrak{V}(T_r, A)$, $r = 1, 2, 3$. This last statement is not explicitly stated in [2] but is implied by the statements

(1) $\mathfrak{V}(T_r, A) = (E_{2r})\int n(p; T_r, A)$, [2, p. 206]; and

(2) $\lim (E_{2r})\int |n(p; T_{nr}, F_n) - n(p; T_r, A)| = 0$, [2, p. 202], which are true under the hypotheses stated above.

The sequence $\{(T_n', F_n)\} = \{(\alpha T_n, F_n)\}$ also has the properties that $\{F_n\}$ invades A, $\rho_n = \sup\{\|T_n'(w) - T'(w)\| : w \in F_n\} = \delta_n$ converges to zero, and $V(T_n', F_n)$ converges to $V(T', A)$. This last statement follows from the fact that area is invariant under rotations,

$V(T_n', F_n) = V(T_n, F_n)$, $V(T', A) = V(T, A)$, [2, p. 335]. Thus from the reasoning above, $\mathfrak{V}(T_{nr}', F_n)$ converges to $\mathfrak{V}(T_r', A)$, $r = 1, 2, 3$.

Then for every integer n, $\|\mathfrak{V}(T',A) - \alpha\mathfrak{V}(T,A)\| \leqq \|\mathfrak{V}(T',A) - \mathfrak{V}(T_n', F_n)\| + \|\mathfrak{V}(T_n', F_n) - \alpha\mathfrak{V}(T_n, F_n)\| + \|\alpha[\mathfrak{V}(T_n, F_n) - \mathfrak{V}(T, A)]\|$. The first and third terms of this expression tend to zero as n tends to infinity. By the first part of our proof, the second term is zero for all n because the image of a boundary curve of a quasilinear mapping is always rectifiable. Thus $\mathfrak{V}(T', A) = \alpha\mathfrak{V}(T, A)$ as stated.

To prove Theorem 4, we will need the result from [2, p. 359], that under the hypotheses of Theorem 3, for every $\epsilon > 0$ there is a set $\mathfrak{S}$ whose indices d, m, μ with respect to (T, A) and indices d', m', μ' with respect to (T', A) are all $< \epsilon$.

Proof of Theorem 4. Evidently $g(p, d)$ is uniformly continuous and bounded on $R' = \{(p,d) : p \in T'(A), \|d\| = 1\}$ and therefore $I(T, A; f)$ and $I(T', A; g)$ both exist by Theorem 1. Let $\{\mathfrak{S}_n\}$ be a sequence of sets of polygons whose indices d_n, m_n, μ_n and d_n', m_n', μ_n' tend to zero. If p_π is any point of $T(\pi)$, $\pi \in \mathfrak{S}_n$, then $\sum_n f(p_\pi, \mathfrak{V}_\pi)$ converges to $I(T, A; f)$ where $\sum_n$ denotes a sum over all $\pi \in \mathfrak{S}_n$. Also $\sum_n g(\alpha p_\pi, \mathfrak{V}_\pi')$ converges to $I(T', A; g)$. But from Theorem 3, $\mathfrak{V}_\pi' = \alpha\mathfrak{V}_\pi$ and thus

$$\sum_n g(\alpha p_\pi, \mathfrak{V}_\pi') = \sum_n g(\alpha p_\pi, \alpha\mathfrak{V}_\pi) = \sum_n f(\alpha^{-1}\alpha p_\pi, \alpha^{-1}\alpha\mathfrak{V}_\pi) = \sum_n f(p_\pi, \mathfrak{V}_\pi).$$

Therefore $I(T, A; f) = I(T', A; g)$.

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