the determinant $|A_{kl}|$ must vanish. This is the announced identity, connecting two different (not linearly dependent) *Heine-Stieltjes* polynomials $P_n^{(\alpha_0, \dots, \alpha_p)}(x)$ and $P_{n'}^{(\alpha_0, \dots, \alpha_p)}(x)$, $n \neq n'$, or $n = n'$, but $(n_1, n_2, \dots, n_p) \neq (n'_1, n'_2, \dots, n'_p)$. For $p=1$ (2) goes over in the known orthogonality relations

$$\int_{-1}^{+1} P_n^{(\alpha_0, \alpha_1)}(x) \cdot P_{n'}^{(\alpha_0, \alpha_1)}(x) |(x+1)^{\alpha_0}(x-1)^{\alpha_1}| = 0$$

with $|(x+1)^{\alpha_0}(x-1)^{\alpha_1}|$ as weight function. We call $|(x-a_0)^{\alpha_0} \dots (x-a_p)^{\alpha_p}|$ still the weight function of the generalized Jacobi polynomials of the order p .

By introduction of the extended *Heine-Stieltjes* polynomials $p_m^{(\alpha_0, \dots, \alpha_p)}(x) = (x-a_0)^{\alpha_0} \dots (x-a_p)^{\alpha_p} P_n^{(\alpha_0, \dots, \alpha_p)}(x)$ and by denoting $\int_{\alpha_k}^{\alpha_k+1} (uvx^l / (x-a_0)^{\alpha_0} \dots (x-a_p)^{\alpha_p}) dx$, by B_{kl} , we obtain $|B_{kl}| = 0$.¹²

We turn finally to the special case, interesting among others for the theory of the transfinite diameter and the *Fekete*-diameters of a finite degree m , that $\alpha_i = \rho_i = 1$, $i=0, \dots, p$. Then the generalized Vandermondean $V_m(\xi_1, \xi_2, \dots, \xi_m; e_1, e_2, \dots, e_m)$ becomes the usual Vandermondean $V_m(\xi_1, \xi_2, \dots, \xi_m) = \prod_{1 \leq i < k \leq m} (\xi_i - \xi_k)$ and, since $A_1(x) = A(x)$, there results $B(x) = A'(x)$.

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¹² u and v are two different extended Heine-Stieltjes polynomials with the same superscripts $\alpha_0, \alpha_1, \dots, \alpha_p$. The weight-function corresponding to them is $|(x-a_0)^{-\alpha_0}(x-a_1)^{-\alpha_1} \dots (x-a_p)^{-\alpha_p}| = |(x-a_0)^{\alpha_0} \dots (x-a_p)^{\alpha_p}|^{-1}$.

AN INTEGRAL TRANSFORMATION RELATION

R. G. BUSCHMAN

In a recent paper Rooney [3] has discussed the interesting formula

$$\int_0^\infty t^{r+1} K_\nu(st) \phi(t^2) dt = 2^{r-1} s^{-r} \int_0^\infty e^{-sy} y^{r-1} f(s^2/4y) dy$$

as a generalization of the formula for the Laplace transformation of $t^r \phi(t^2)$. Here $K_\nu(x)$ is the modified Bessel function of the second kind and

$$f(s) = \int_0^\infty e^{-st} \phi(t) dt = \mathcal{L}\{\phi(t)\}.$$

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In this paper we generalize further the relations for the integral transform of $k(t)F[g(t)]$ obtained in [1] and then show that the above formula can be obtained by choosing specific functions.

Suppose that for some kernel $K(s, t)$

$$\int_0^\infty K(s, t)k(t)F[g(t)]dt = \mathfrak{K}\{k(t)F[g(t)]\}$$

exists. If g and its inverse function $h=g^{-1}$ are monotonic, differentiable, and real on $(0, \infty)$ and $g(0)=0$, $g(\infty)=\infty$ (or $g(0)=\infty$, $g(\infty)=0$), then

$$\mathfrak{K}\{k(t)F[g(t)]\} = \int_0^\infty K[s, h(p)]k[h(p)]|h'(p)|F(p)dp.$$

Next we consider a symmetric kernel $S(u, p) \equiv S(p, u)$. If there is a function $\Phi(s, u)$ such that

$$\phi(s, p) = \int_0^\infty S(u, p)\Phi(s, u)du = \mathfrak{s}\{\Phi(s, u)\}$$

where

$$\phi(s, p) = K[s, h(p)]k[h(p)]|h'(p)|,$$

then we obtain

$$\mathfrak{K}\{k(t)F[g(t)]\} = \int_0^\infty \left(\int_0^\infty S(u, p)\Phi(s, u)du \right) F(p)dp.$$

Now, provided that the function $F(p)$ is such that the interchange of integration is valid and that $\mathfrak{s}\{F(p)\} = f(u)$, then

$$\mathfrak{K}\{k(t)F[g(t)]\} = \int_0^\infty \Phi(s, u)f(u)du.$$

As an example, the formula of Rooney can now be obtained by letting $K(s, t) = K_\nu(st)$, $S(u, p) = e^{-up}$, $F(t) = \phi(t)$, $g(t) = t^2$, $k(t) = t^{\nu+1}$, and $\Phi(s, u)$ can then be obtained from tables [2, 5.16(40)].

The known relation for the Laplace transformation

$$\mathfrak{L}\{t^{\nu-1}F(1/t)\} = \int_0^\infty (u/s)^{\nu/2} J_\nu[2(su)^{1/2}]f(u)du,$$

where $\mathfrak{L}\{F(t)\} = f(s)$, is obtained by choosing $K(s, t) = e^{-st}$, $S(u, p) = e^{-up}$, $g(t) = t^{-1}$, and $k(t) = t^{\nu-1}$. $\Phi(s, u)$ is then obtained from [2, 5.6(40)].

For another example consider the parabolic cylinder function and let $K(s, t) = t^{2\nu} e^{s^2 t^2/4} D_{-2\nu}(st)$, $S(u, p) = e^{-up}$, $g(t) = t^{-2}$, and $k(t) = t^{-3}$. $\Phi(s, u)$ is then obtained from [2, 5.18(7)] so that after some simplification we have

$$\int_0^\infty t^{2\nu-3} e^{s^2 t^2/4} D_{-2\nu}(st) F(t^{-2}) dt = [2\Gamma(2\nu)]^{-1} \int_0^\infty u^{2\nu-1} e^{-su} f(u^2/2) du,$$

in which $\mathfrak{L}\{F(t)\} = f(u)$. In particular if $F(t) = t^{-3/2} e^{-b/4t}$, then $f(u) = 2\pi^{1/2} b^{-1/2} e^{-b^{1/2} u^{1/2}}$ and

$$\begin{aligned} \int_0^\infty e^{-bt^2/4} t^{2\nu} e^{s^2 t^2/4} D_{-2\nu}(st) dt \\ = \pi^{1/2} [b^{1/2} \Gamma(2\nu)]^{-1} \int_0^\infty u^{2\nu-1} \exp\{-[s + (b/2)^{1/2}]u\} du \\ = \pi^{1/2} b^{-1/2} [s + (b/2)^{1/2}]^{-2} \end{aligned}$$

for $\nu > 0$, $b > 0$, $s > 0$.

To summarize, under the conditions stated above, if

$$\mathfrak{S}\{\Phi(s, u)\} = \phi(s, p) \quad \text{and} \quad \mathfrak{S}\{F(u)\} = f(p)$$

where

$$\phi(s, p) = K[s, h(p)] k[h(p)] |h'(p)|,$$

then

$$\mathfrak{K}\{k(t)F[g(t)]\} = \int_0^\infty \Phi(s, u) f(u) du.$$

REFERENCES

1. R. G. Buschman, *A substitution theorem for the Laplace transformation and its generalization to transformations with symmetric kernel*, Pacific J. Math. vol. 7 (1957) pp. 1529-1533.
2. A. Erdélyi et al., *Tables of integral transforms*, New York, 1954.
3. P. G. Rooney, *A property of the Laplace transformation*, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 883-886.

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