CONVEX SETS AND NEAREST POINTS. II[1]

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1. **Introduction.** In 1935 Motzkin [5] showed that if S is a subset of the euclidean plane E and $z \in S$ then the set S_z of all points in E having z as a nearest point in S is closed and convex. In [7] this result was easily extended to inner product spaces of arbitrary dimension. In §2 of the present paper we show that *any* closed convex subset T of a complete inner product space E can be realized in such a manner, i.e. there exists a set S and a point $z \in S$ such that $T = S_z$. Further, it is true that if E is a normed linear space of dimension at least three then every closed convex subset T can be so realized *only* if E is a complete inner product space.

If A is a subset of a normed linear space E and x, y are in E we say that y is *point-wise closer* to A than is x provided $\|y-a\| < \|x-a\|$ for each $a \in A$. If x is such that no point of E is point-wise closer to A than is x we call x a *closest-point* to A. Fejér [1] has noted that in the euclidean plane the set $C(A)$ of all closest points to A is precisely $K(A)$, the closed convex hull of A. In applying this result, Fejér, and later Motzkin and Schoenberg [6], actually used a weaker version, which we will call *property* (F): If $A \subset E$, then $C(A) \subset K(A)$. In §3 we show that property (F) characterizes complete inner product spaces of dimension at least three, while in two-dimensional spaces it is equivalent to strict convexity. We also extend Fejér's characterization of $K(A)$ to complete inner product spaces and show that it holds in strictly convex two-dimensional spaces if A is bounded.

2. **Nearest-point sets.** If S is a subset of a normed linear space E and if $z \in S$, S_z will denote the set $\{x: \|x-z\| = \inf_{y \in S} \|x-y\|\}$ of all points in E whose distance from S is attained at z. Using the alternative formulation $S_z = \{x: \|x-z\| \leqq \|x-y\|$ for each $y \in S\}$ it is easy to see that S_z is always closed. In several papers [2; 3; 4] James has successfully exploited a concept of orthogonality which is defined as follows: We say that *x is orthogonal to y* (and write $x \perp y$) if $\|x\| \leqq \|x - \lambda y\|$ for each $\lambda \in R$, where R is the real numbers. Note that this is equivalent to saying that x has the origin ϕ as a nearest point

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in $Ry=\{\lambda y:\lambda\in R\}$, the line determined by y and ϕ (assuming $y\neq\phi$)—in other words, $x\in(Ry)_\phi$. That this is a generalization of the usual inner product space concept of orthogonality is easily verified, using the identity

$$\|x-\lambda y\|^2=\|x\|^2-2\lambda(x,y)+\|y\|^2 \tag{1}$$

by merely showing that in such a space $x\perp y$ if and only if $(x, y)=0$. If L is a linear subset of E we write $L\perp y(y\perp L)$ provided $x\perp y(y\perp x)$ for each $x\in L$, and if M is linear, $L\perp M$ is defined in a similar way. Note that $x\perp y$ if and only if $Rx\perp Ry$.

James has proved a theorem [**4**, Theorem 4] which, after a slight extension, will be quite useful to us.

THEOREM 1 (JAMES). *Suppose E is a normed linear space of dimension at least three. Then E is a complete inner product space provided each hyperplane through the origin is orthogonal to some point $x\neq\phi$.*

PROOF. James has shown that the condition is sufficient to imply that E is an inner product space, so we need only show that E is also complete. Suppose $y\in F$, where F is the completion of E. Then the set $H=\{z: z\in F \text{ and } (z, y)=0\}$ is a hyperplane in F and $H_0=H\cap E$ is a hyperplane in E which contains ϕ. By hypothesis there exists $x\neq\phi$ such that $H_0\perp x$. But, as noted previously, this implies that $(z, x)=0$ for each $z\in H_0$. Hence, by continuity of the inner product and the fact that H_0 is dense in H, we have $(z, x)=0$ for each $z\in H$. This implies that $y\in Rx\subset E$, so that $E=F$ is complete.

Note that if H is a hyperplane in an inner product space and x is a point of least norm in H then $x\in(H-x)_\phi$ or $x\perp H-x$.

We call a subset T of a normed linear space E a *nearest-point* set if there exists a set $S\subset E$ and a point $z\in S$ such that $T=S_z$. In [**7**] it was shown that every nearest-point set in E is closed and convex if and only if E is an inner product space. Here we consider the question: For what class of normed linear spaces is every closed convex set a nearest-point set? This is partially answered by the following theorem.

THEOREM 2. *In a complete inner product space E every closed convex set T is a nearest-point set.*

PROOF. If $T=E$, let $S=\{\phi\}$. Otherwise, $T=\bigcap_{H\in\mathcal{H}} H'$ where $\mathcal{H}$ is the collection of all hyperplanes H such that H determines a closed half-space H' for which $T\subset H'$. Pick any point of T—we will suppose that it is the origin. For $H\in\mathcal{H}$ let x_H be the point of least norm in H and define $S=\{2x_H: H\in\mathcal{H}\}\cup\{\phi\}$. Since $x_H\perp H-x_H$ for each

$H\in\mathcal{H}$, we have $H'-x_H=\{y: (x_H, y)\leqq 0\}$ or $H'=\{y: (x_H, y-x_H)\leqq 0\}$. But, letting $z=y-x_H$, $(z, x_H)\leqq 0$ if and only if $2(z, x_H)\leqq -2(z, x_H)$ which is in turn equivalent to $\|z\|^2+2(z, x_H)+\|x_H\|^2\leqq\|z\|^2-2(z, x_H)+\|x_H\|^2$. This, however, is the same as $\|z+x_H\|^2\leqq\|z-x_H\|^2$ or $\|y\|^2\leqq\|y-2x_H\|^2$. Thus, $H'=\{y: \|y\|\leqq\|y-2x_H\|\}$ and hence $T=\bigcap_{H\in\mathcal{H}} H'=\{y: \|y\|\leqq\|y-2x_H\|$ for each $H\in\mathcal{H}\}=\{y: \|y\|\leqq\|y-z\|$ for each $z\in S\}=S_\phi$.

The above proof shows that for *any* $z\in T$ there exists a set $S(T, z)$ such that $T=S(T, z)_z$. Applying the (constructive) method of the proof to a specific example we find, with the aid of some elementary analytic geometry, that if T is the closed disc bounded by the curve $r=2\cos\theta$ then $S(T, \phi)$ is the area not inside the cardioid $r=2(1+\cos\theta)$. Note that we could omit from S any point not on the cardioid and still have $T=S_\phi$. We can, however, show that in a certain sense the set S can be taken to be unique. Suppose that T is a nearest-point subset of a normed linear space E; then there exists $S\subset E$ and $z\in S$ such that $T=S_z$. Let $\mathcal{S}$ be the collection of all sets S containing z such that $S_z=T$. Then, letting $Q(T, z)=\bigcup_{S\in\mathcal{S}} S$, we see that $Q(T, z)_z=\bigcap_{S\in\mathcal{S}} S_z=T$ so that $Q(T, z)\in\mathcal{S}$ and is the biggest member of $\mathcal{S}$. Henceforth, if a set T is a nearest-point set, $Q(T, z)$ will denote the set defined above. If there is no chance for confusion we will simply denote it by Q.

Suppose that $T=S_z$ for some S and $z\in S$. Then $Q(T, z)=\{x: \|x-y\|\geqq\|z-y\|$ for each $y\in T\}$. For clearly $x\in Q$ implies $\|x-y\|\geqq\|z-y\|$ for each $y\in T$; and if this latter is true, $T\subset[Q\cup\{x\}]_z\subset Q_z=T$ so that $[Q\cup\{x\}]_z=T$ and by the maximality property of Q, $x\in Q\cup\{x\}\subset Q$. Using this description of Q it is easy to see that Q is closed. Further, the set S constructed in the proof of Theorem 2 is actually equal to $Q(T, \phi)$, for $T=S_\phi$ implies that $S\subset\{x: \|x-y\|\geqq\|y\|$ for each $y\in T\}$. Suppose that x is a point of the latter set, then the hyperplane $H=\{w: \|w-x\|=\|w\|\}$ passes through $(1/2)x$ and is orthogonal to Rx. Hence, if H' is the closed half-space determined by H which contains ϕ, $y\in T$ implies $\|y\|\leqq\|y-x\|$ so that $y\in H'$ and thus $T\subset H'$. We have, then, that $H\in\mathcal{H}$ and that x is of the form $2x_H$ and is therefore in S.

It is natural to wonder if Q may be convex. Suppose that $T=Q_\phi$ and that Q *is* convex, then by Lemmas 3.1 and 4.1 of [7] T must be a convex cone with vertex ϕ. Using identity (1) it is easily verified that whenever T is a cone with vertex ϕ, $Q=\{x: (x, y)\leqq 0$ for each $y\in T\}$ and hence Q is also a convex cone with vertex ϕ (the *dual* or *polar* cone of T [8]). Thus, Q is convex if and only if both T and Q are convex cones with common vertex.

In what follows we show that the hypothesis in Theorem 2 that E be a complete inner product space is necessary, if the dimension of E is at least three.

THEOREM 3. *Suppose that E is a normed linear space of three or more dimensions and that every closed convex subset of E is a nearest-point set. Then E is a complete inner product space.*[3]

PROOF. Suppose H is a hyperplane in E which passes through the origin. By Theorem 1 we can conclude that E is a complete inner product space provided every such hyperplane is orthogonal to some $x \neq \phi$, i.e. provided $H \perp Rx$ for some $x \neq \phi$. Now, H is closed and convex so there exists a set S and a point $z \in S$ such that $H = S_z$. By the preceding discussion there exists a maximal set $Q(H, z)$ such that $H = Q_z$. If $Q = \{z\}$, $Q_z = H$ would be the entire space, contradicting the fact that H is a hyperplane. Pick $x \in Q \sim \{z\}$ and suppose $y \in H$ and $\lambda \in R \sim \{0\}$. Since $z \in H$, there exists $w \in H$ such that $y = \lambda(w - z)$. We know that $\|w - x\| \geqq \|w - z\|$ hence $\|\lambda(w - x)\| \geqq \|\lambda(w - z)\|$ or $\|y - \lambda(x - z)\| \geqq \|y\|$. Therefore $H \perp R(x - z)$, which was to be shown.

What if E is two-dimensional and every closed convex subset of E is a nearest-point set? By somewhat lengthy arguments we have been able to show that this implies strict convexity of E, and that E is an inner product space provided we assume that E is smooth. We have been unable to show that the assumption of smoothness is necessary. This result indicates, however, that strict convexity alone does not imply that every closed convex subset of E is a nearest-point set, for such an implication would in turn imply the false statement that every smooth and strictly convex two-dimensional normed linear space is an inner product space.

3. **Convex sets and closest points.** We first prove Fejér's theorem (see introduction) in a more general setting.

THEOREM 4 (FEJÉR). *If A is a subset of a complete inner product space E, the closed convex hull $K(A)$ of A is the set $C(A)$ of closest points to A.*

PROOF. Suppose $x \notin K(A)$, then there exists a hyperplane H separating x from $K(A)$ such that $x \notin H$. Since E is complete we know that the distance from x to H is attained at some (unique) point of H. Without loss of generality we may assume that this point is the origin, so that $\|x\| \leqq \|x - z\|$ for each $z \in H$. This says, then, that $x \perp H$ and hence $H \perp x$. Thus, if $a \in A$, there exists $z \in H \cap [a, x]$. Now,

[3] We wish to thank the referee for suggesting changes which simplified the proof of this theorem, as well as the proof of Theorem 5.

$\|z\| < \|z-x\|$ and therefore $\|a\| \leqq \|a-z\| + \|z\| < \|a-z\| + \|z-x\| = \|a-x\|$. Hence ϕ is point-wise closer to A than is x, giving $x \notin C(A)$.

Suppose $x \notin C(A)$, then there exists $y \in E$ which is point-wise closer to A than is x. The set $H = \{z: \|z-y\| = \|z-x\|\}$ is a hyperplane through $(1/2)(x+y)$ which determines a closed half-space $H' = \{z: \|z-y\| \leqq \|z-x\|\}$ containing A. Hence $K(A) \subset H'$ and since $x \notin H'$, $x \notin K(A)$.

Defining property (F) as in the introduction, we now determine the class of spaces possessing the property.

THEOREM 5. *Suppose that E is a normed linear space of dimension at least three which possess property* (F). *Then E is a complete inner product space.*

PROOF. We again make use of Theorem 1. Suppose that H is a hyperplane through ϕ. If $x \notin H = K(H)$ then by assumption $x \notin C(H)$ and there must exist a point y such that $\|h-y\| < \|h-x\|$ for each $h \in H$. The line L determined by x and y must hit H at w, say. (For ϕ has a nearest point z in L, so if L were parallel to H, $h = x-z$ would be in H and have x as a nearest-point in L. In particular, h would not be closer to $y \in L$ than to x, a contradiction.) Now, $w = \alpha y + (1-\alpha)x$ for some $\alpha \neq 0$, so if $\lambda \neq 0$ and h is any point of H we have $\alpha\lambda^{-1}h + w \in H$ and $\|h - \lambda(x - y)\| = \|\lambda\alpha^{-1}[(\alpha\lambda^{-1}h + w) - x]\| > \|\lambda\alpha^{-1}[(\alpha\lambda^{-1}h + w) - y]\| = \|h + \beta\lambda(x-y)\|$, where we have let $\beta = \alpha^{-1}(1-\alpha)$. If $\beta = 0$ we have the inequality we are seeking; otherwise, we use the fact that $h \in H$ if and only if $\beta^{-1}h \in H$ to conclude, by induction, that $\|h - \lambda(x-y)\| > \|h + \beta^n\lambda(x-y)\|$ for each positive integer n. Setting $h = \phi$ shows that $|\beta| < 1$, so, taking the limit as $n \to \infty$, we have $\|h - \lambda(x-y)\| > \|h\|$. This shows that $H \perp (x-y)$ and hence we can conclude from Theorem 1 that E is a complete inner product space.

LEMMA 6. *If a normed linear space E has property* (F) *then E is strictly convex.*

PROOF. If E is not strictly convex there exist points x, y in E such that the distance from y to each point of the segment $[-x, x]$ is exactly 1. Let $A = [-y, y]$; then $x \notin A = K(A)$ and hence $x \notin C(A)$, by hypothesis. There must exist a point z in E which is point-wise closer to A than is x. Thus, $\|z-y\| < \|x-y\| = 1$ and $\|z+y\| < \|x+y\| = 1$ so $2 = \|2y\| = \|(-y) - y\| \leqq \|-y-z\| + \|z-y\| < 2$, a contradiction.

THEOREM 7. *Suppose that E is a two-dimensional normed linear space. Then E possesses property* (F) *if and only if it is strictly convex.*

PROOF. Lemma 6 proves the necessity. Suppose E is strictly convex and $A \subset E$. If $x \notin K(A)$ there exists a line L separating $K(A)$ from x such that $x \notin L$. We can suppose that $\phi \in L$ and that $L = Ry$ for some $y \neq \phi$. There exists a line M through ϕ such that $y \perp M$ [3, Theorem 2.2], so $M+x$ hits Ry at αy for some $\alpha \in R$. We will show that αy is point-wise closer to A than is x. Suppose $a \in A$, then $[a, x]$ hits Ry at βy, say. Now $M+x = M+\alpha y$ so $x - \alpha y \in M \sim \{\phi\}$ and hence, by strict convexity, $\|\beta y - \alpha y\| < \|(\beta y - \alpha y) - (x - \alpha y)\| = \|\beta y - x\|$. Then

$$\|a - \alpha y\| \leqq \|a - \beta y\| + \|\beta y - \alpha y\| < \|a - \beta y\| + \|\beta y - x\| = \|a - x\|.$$

Hence $x \notin C(A)$ and we have $C(A) \subset K(A)$.

In stating his theorem Fejér assumed that A was compact. This is an unnecessary restriction if we wish to prove his characterization of $K(A)$ in the euclidean plane, but it is interesting to note that if we merely assume that A is *bounded* we can prove his characterization in any two-dimensional strictly convex normed linear space.

THEOREM 8. *Suppose that A is a bounded subset of a strictly convex two-dimensional normed linear space E. Then $K(A) = C(A)$.*

PROOF. That $C(A) \subset K(A)$ follows from the strict convexity and Theorem 7. Suppose that $x \notin C(A)$, then there exists a point $y \in E$ such that $\|y - a\| < \|x - a\|$ for each $a \in A$. We can suppose without loss of generality that $x = \phi$. Now there exists a point $z \neq \phi$ such that $z \perp y$ [3] and hence $L \perp y$ for $L = Rz$. Letting L' be the closed half-space determined by L which does not contain y, we see that cl A, the closure of A, is contained in $E \sim L'$ as follows: Let $S = \{w: \|w - y\| \leqq \|w\|\}$; then S is closed, contains A and hence contains cl A. Furthermore, if w is a point of L', we can use strict convexity and the triangle inequality (as in the proof of Theorem 7) to show that $\|w\| < \|w - y\|$. Thus, $w \notin S$ so $S \subset E \sim L'$ and cl $A \subset E \sim L'$.

Since A is bounded, cl A is compact and hence conv cl A, the convex hull of cl A, is closed. Thus, $K(\text{cl } A) = \text{conv cl } A$ and since $E \sim L'$ is convex, conv cl $A \subset E \sim L'$. We have, then, that $K(A) \subset K(\text{cl } A) \subset E \sim L'$, so that ϕ, being in $L \subset L'$, is not in $K(A)$. Therefore $C(A) = K(A)$ for any bounded set A.

It is possible to construct an example in which the above result fails for an unbounded set A.

BIBLIOGRAPHY

1. L. Fejér, *Über die Lage der Nullstellen von Polynomen, die aus Minimumforderungen gewisser Art entspringen*, Math. Ann. vol. 85 (1922) pp. 41–48.

2. R. C. James, *Orthogonality in normed linear spaces*, Duke Math. J. vol. 12 (1945) pp. 291–302.

3. ———, *Orthogonality and linear functionals*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 265–292.

4. ———, *Inner products in normed linear spaces*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 559–566.

5. Th. Motzkin, *Sur quelques propriétés caractéristiques des ensembles bornés non convexes*, Atti Accad. Naz. Lincei. Rend. 6 vol. 21 (1935) pp. 773–779.

6. Th. Motzkin and I. J. Schoenberg, *The relaxation method for linear inequalities*, Canad. J. Math. vol. 6 (1954) pp. 393–404.

7. R. R. Phelps, *Convex sets and nearest points*, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 790–797.

8. L. Sandgren, *On convex cones*, Math. Scand. vol. 2 (1954) pp. 19–28.

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NOTE ON PRODUCTS IN Ext

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The objective of this note is to present an interrelation between the $\vee$-product in [1] and the composition product in [2; 3], which in turn gives a comparison of the cup-product with the composition product. Similar relations can be obtained also for other products involving Tor and (iterated) connecting homomorphisms.

We retain the notations in [1, Chapter XI]. The external product

$$\vee\colon \operatorname{Ext}_{\Lambda\otimes\Sigma^*}(A, C) \otimes \operatorname{Ext}_{\Sigma\otimes\Gamma^*}(A', C') \to \operatorname{Ext}_{\Lambda\otimes\Gamma^*}(A \otimes_\Sigma A', C \otimes_\Sigma C')$$

is defined in the situation $({}_\Lambda A_\Sigma, {}_\Lambda C_\Sigma, {}_\Sigma A'_\Gamma, {}_\Sigma C'_\Gamma)$ under the following assumption: (i) Λ, Γ, Σ are K-projective; (ii) $\operatorname{Tor}_n^\Sigma(A, A')=0$ for $n>0$. To this situation we now add $({}_\Lambda B_\Sigma, {}_\Sigma B'_\Gamma)$ and (ii′) $\operatorname{Tor}_n^\Sigma(B, B')=0$ for $n>0$. For $a\in\operatorname{Ext}^p_{\Lambda\otimes\Sigma^*}(A, B)$ and $b\in\operatorname{Ext}^q_{\Lambda\otimes\Sigma^*}(B, C)$ the composition product $b\circ a$ lies in $\operatorname{Ext}^{p+q}_{\Lambda\otimes\Sigma^*}(A, C)$. For $a'\in\operatorname{Ext}^{p'}_{\Sigma\otimes\Gamma^*}(A', B')$ and $b'\in\operatorname{Ext}^{q'}_{\Sigma\otimes\Gamma^*}(B', C')$, $b'\circ a'$ lies in $\operatorname{Ext}^{p'+q'}_{\Sigma\otimes\Gamma^*}(A', C')$. ($b\circ a = a \bigcirc b$ in the notation of [2].)

PROPOSITION 1. $(b\circ a)\vee(b'\circ a') = (-1)^{pq'}(b\vee b')\circ(a\vee a')$.

In fact let X, Y be $\Lambda\otimes\Sigma^*$-projective resolutions of A, B respectively, and let X', Y' be $\Sigma\otimes\Gamma^*$-projective resolutions of A', B'. Then $X\otimes_\Sigma X'$, $Y\otimes_\Sigma Y'$ are $\Lambda\otimes\Gamma^*$-projective resolutions of $A\otimes_\Sigma A'$, $B\otimes_\Sigma B'$. We consider these resolutions as chain complexes with 0's in negative dimensions. Suppose that a, b, a', b' are respectively represented by maps $\alpha\colon X_p\to B$, $\beta\colon Y_q\to C$, $\alpha'\colon X'_{p'}\to B'$, and $\beta'\colon Y'_{q'}\to C'$. The map α

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