PRODUCT SPACES IN n-MANIFOLDS

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The following theorem for product spaces appears not to have been noticed.

THEOREM. *Let* $A_1, A_2, \cdots, A_n$ *be (nondegenerate) locally compact connected sets. Suppose* $P = A_1 \times A_2 \times \cdots \times A_n$ *can be imbedded in an n-manifold,* M^n. *Then each* A_i *is either an arc, a simple closed curve, a ray, or an open curve.*

PROOF. The product of a k-dimensional locally compact set by any locally compact connected set has dimension at least $k+1$ [**1**]. Since each A_i has dimension at least 1, and since P has dimension at most n, it follows that each A_i has dimension 1, and that P has dimension n.

Suppose that, for some k, the set A_k is not locally connected. Then there exist an open subset U of A_k, a continuum B in $\overline{U}$, and a sequence $\{B_j\}$ of distinct components of U having B as its limiting set. Now $P' = B \times \prod_{i \neq k} A_i$ has dimension n, and hence has interior points [**2**, p. 44]. But each point of P' is a limit point of $\bigcup_j (B_j \times \prod_{i \neq k} A_i)$, which is a contradiction. Hence for each i $(1 \leqq i \leqq n)$, A_i is locally connected.

By [**3**, p. 428], each A_i is either one of the simple continuous curves in the conclusion of the theorem, or contains a simple triod, i.e., the union of two arcs, X and Y, intersecting in just one point and forming a set homeomorphic to a letter T. Suppose that A_1 contains such a triod, $X \cup Y$. Let X_j be an arc in A_j, $j \neq 1$. Then $(X \cup Y) \times \prod_{j=2}^{n-1} X_j$ contains a T_{n-1}-set, T, that is, a set homeomorphic to the union of an $(n-1)$-cell and an arc intersecting the cell in a point which is in the combinatorial interior of the $(n-1)$-cell and which is an end point of the arc. The set $T \times X_n$ is the union of uncountably many disjoint T_{n-1}-sets, and is a subset of $\prod_{i=1}^{n} A_i$; but M^n cannot contain uncountably many disjoint T_{n-1}-sets [**5**]. It follows that no A_i contains a triod, which completes the proof.

Unless M^n itself is an n-torus, and $M^n = P$, not all the sets A_i can be simple closed curves. Every other possible combination can occur, however.

It will be noted that actually the only property of M^n that was

Presented to the Society, October 25, 1958; received by the editors August 19, 1958.

[1] Part of the work on this was done while the first author was an NSF Fellow and the second author was an Esso Fellow.

used in the proof of local connectedness was the fact that n-dimensional subsets of M^n have interior points. Frank Raymond has proved recently [4] that an n-dimensional generalized manifold in the sense of Wilder has this property. Further the argument in [5] is valid in generalized manifolds, so that our result holds for these also.

If a product space P is a product of 1- and 2-dimensional continua, and is a manifold (with or without boundary) then so are the factors [6]. But if we say instead that the sum of the dimensions is n and that P is imbeddable in E^n, it need not follow that each factor is locally connected. The plane, for example, contains 2-dimensional Cantorian manifolds that are not locally connected, and the product of one of these by an interval would be such an example in E^3.

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