APPROXIMATION BY BOUNDED ANALYTIC FUNCTIONS: GENERAL CONFIGURATIONS¹

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A recent paper [1] presents some results on approximation by bounded analytic functions (Problem β), relatively general regarding the continuity properties of the functions approximated, but not as general as one might desire regarding the topological properties of the point sets admitted. It is our object here to establish a new result (Theorem 1, below) which is topologically more general and includes all the previous results on degree of approximation (Problem β) when the approximated functions are given, and new ones as well. In the proof we rely strongly on [1].

We mention several simple examples not previously included which are included in the present Theorem 1; the emphasis is of course on the topology rather than on the specific point sets or numerical values. In each case we study approximation to a given function $f(z)$ on a closed set E by functions $f_n(z)$ analytic and bounded in the interior points of a set $D+E$; here $f(z)$ is assumed analytic in the interior points of E , in suitable adjacent subregions D_ρ of D , and also on the common boundary of E and D_ρ ; moreover $f(z)$ is assumed to satisfy certain continuity conditions on the boundary of D_ρ in D .

EXAMPLE 1.

$$E: 1 \leq |z| \leq 2, \quad D: 2 < |z| < 3, \quad D_\rho: 2 < |z| < \rho, \quad 2 < \rho < 3.$$

EXAMPLE 2.

$$\begin{aligned} E: |z| &= 3, & D: 1 < |z| < 3 \text{ plus } 3 < |z| < 5, \\ D_\rho: \rho_1 < |z| < 3 \text{ plus } 3 < |z| < \rho_2, & 1 < \rho_1 < 3 < \rho_2 < 5. \end{aligned}$$

EXAMPLE 3.

$$\begin{aligned} E: 2 \leq |z| \leq 4, & \quad D: 1 < |z| < 2 \text{ plus } 4 < |z| < 5, \\ D_\rho: \rho_1 < |z| < 2 \text{ plus } 4 < |z| < \rho_2, & 1 < \rho_1 < 2, 4 < \rho_2 < 5. \end{aligned}$$

Our main result is (in the extended plane)

THEOREM 1. *Let $B_1, B_2, \dots, B_\mu, C_1, C_2, \dots, C_\nu$ be a set of mutually disjoint analytic Jordan curves; we denote by B the sum of the B_j and*

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by C the sum of the C_j . Let D be a bounded open set whose boundary is all of C plus a (not necessarily proper) subset of the B_j , and indeed let each maximal (i.e. contained in no larger) subregion of D be bounded by a nonempty subset of the B_j plus a nonempty subset of C . Let C not separate D . Let E be a closed set disjoint from D whose boundary is B ; but E need not be the closure of an open set. Let the set $D+E$ be connected.

Let $u(z)$ be the function harmonic in D , continuous in the closure of D , equal to zero on B and unity on C . Let Γ_σ , $0 < \sigma < 1$, denote generically the locus $u(z) = \sigma$ in D , and let D_σ denote the set $0 < u(z) < \sigma$ in D .

Let the function $f(z)$ be analytic in the interior points of E , continuous on E , and also analytic in D_ρ , where Γ_ρ has no multiple points; thus $f(z)$ is analytic also in the interior points of $E+D_\rho$. Let $f(z)$ be of class $L(p, \alpha)$ for $0 < \alpha < 1$ or Z_p for $\alpha = 1$ on Γ_p .

Then there exist functions $f_n(z)$ analytic in the interior points of $D+E$ and continuous on E , satisfying the inequalities ($n=1, 2, \dots$)

$$\begin{aligned} (1) \quad & |f(z) - f_n(z)| \leq A_1 e^{-n\rho} / n^{p+\alpha}, \quad z \text{ on } E, \\ (2) \quad & |f_n(z)| \leq A_2 e^{n(1-\rho)} / n^{p+\alpha}, \quad z \text{ in } D + E. \end{aligned}$$

Here and below the letter A with subscript indicates a constant independent of n and z .

We have assumed that B is the boundary of E because a curve B_j consisting wholly of interior points of E would serve no useful purpose. We have assumed that C does not separate D , for if C_j separates D it also separates E ; the set $D+E$ is not connected, and if the hypothesis of Theorem 1 is otherwise satisfied, approximation by functions $f_n(z)$ of the kind required reduces to two or more separate problems each included in Theorem 1 as stated.

The classes $L(p, \alpha)$ and Z_p are taken as defined in [1], with p an integer (positive, negative, or zero), and with $0 < \alpha \leq 1$.

For the proof we assume $D+E$ to be bounded, which can be arranged by a linear transformation if necessary. We assume also $p \geq 0$, but this is merely for convenience in exposition to avoid long circumlocutions; with $p \geq 0$ the function $f(z)$ is continuous on Γ_ρ , so Cauchy integrals for $f(z)$ and z in D_ρ may be taken over Γ_ρ ; with $p < 0$ the function $f(z)$ may become infinite as z approaches Γ_ρ , so Cauchy integrals for $f(z)$ cannot be taken over Γ_ρ , but nevertheless can be taken over variable auxiliary Jordan curves in D_ρ near Γ_ρ depending on z .

Let $D^{(i)}$ be any maximal subregion of D , and let $\Gamma^{(i)}$ be a component of Γ_ρ in $D^{(i)}$. Let the maximal subregion d_ρ of D_ρ bounded in part by $\Gamma^{(i)}$ be bounded by $\Gamma^{(i)}$ plus the subset $B^{(i)}$ of B . We set

$$(3) \quad \phi_1(z) \equiv \frac{1}{2\pi i} \int_{\Gamma^{(j)}} \frac{f(t)dt}{t-z},$$

where the integral is taken over $\Gamma^{(j)}$ in the positive sense with respect to d_ρ . The function $\phi_1(z)$ defined by (3) is analytic in the extended plane not merely in d_ρ but throughout the Jordan region bounded by $\Gamma^{(j)}$ and containing d_ρ , and throughout the Jordan region bounded by $\Gamma^{(j)}$ not containing d_ρ . We set also

$$(4) \quad \phi_2(z) \equiv \frac{1}{2\pi i} \int_{B_k} \frac{f(t)dt}{t-z},$$

where the integral is taken over a component B_k of $B^{(j)}$ in the positive sense with respect to d_ρ . The function $\phi_2(z)$ is analytic throughout d_ρ , and in d_ρ we have

$$(5) \quad f(z) \equiv \phi_1(z) + \sum_k \phi_2(z),$$

so $\phi_2(z)$ is continuous even on each B_k when defined there by (5), and $\phi_1(z)$ is of class $L(p, \alpha)$ or Z_p on $\Gamma^{(j)}$ when defined there by (5).

The function $f(z)$ is analytic in the interior points of the set $E+D_\rho$, continuous in the closure of that set. On each component K (necessarily a closed bounded region) of the closure of $E+D_\rho$ the Cauchy integral of types (3) and (4) for $f(z)$ over the boundary of K represents $f(z)$ in the interior points of K and represents zero on all other components of $E+D_\rho$. Thus for z in the interior points of K , the function $f(z)$ is represented both by its Cauchy integral over the boundary of K and its Cauchy integral over the boundary of $E+D_\rho$. Each component of $f(z)$, that is to say a function $\phi_1(z)$ or $\phi_2(z)$ defined by (3) or (4), represents in each of the two regions bounded by $\Gamma^{(j)}$ or B_k an analytic function which (when suitably defined) is continuous on the boundary, since $f(z)$ is continuous on E and on Γ_ρ . We shall have occasion to use the function $\phi_2(z)$ defined by (4) only over a curve B_k not separating $D+E$, and for z in the closure of the Jordan region bounded by B_k containing $D+E$. We consistently define the component $\phi_1(z)$ in (3) on $\Gamma^{(j)}$ itself so that $\phi_1(z)$ shall be continuous in the one-sided closed neighborhood of $\Gamma^{(j)}$ containing points of D_ρ . Similarly we define the component $\phi_2(z)$ in (4) on B_k itself so that $\phi_2(z)$ is continuous in the one-sided closed neighborhood of B_k containing interior points of $D+E$.

By a fundamental theorem on approximation [1, Theorem 6] it now follows that we can approximate in the manner of (1) and (2) to the function $\phi_1(z)$ defined by (3) both on $B^{(j)}$ and throughout E ,

even exterior to $D^{(i)}$, by functions $f_{nj}(z)$ analytic and bounded not merely in $D^{(i)}$ but also throughout $D+E$. Only a part of these properties of $f_{nj}(z)$ is expressed in

$$(6) \quad |\phi_1(z) - f_{nj}(z)| \leq A_3 e^{-n\rho} / n^{p+\alpha}, \quad z \text{ on } E,$$

$$(7) \quad |f_{nj}(z)| \leq A_4 e^{n(1-\rho)} / n^{p+\alpha}, \quad z \text{ on } D+E.$$

The inequalities (6) and (7) may be valid throughout larger point sets than those just mentioned. Thus $\phi_1(z)$ is analytic and (6) is valid throughout the closure E' of the point set which is separated by B from $\Gamma^{(i)}$, and (7) is valid on $D+E'$.

Each maximal subregion $D^{(i)}$ of D may be treated by this method, and each subregion d_ρ of D_ρ . We obtain various functions $\phi_1(z)$ and $\phi_2(z)$ by the analogues of (3) and (4), and we obtain for each $\phi_1(z)$ analogues of the inequalities (6) and (7).

To examine in more detail the Cauchy integral representation of $f(z)$ for z on $E+D_\rho$, we categorize the curves B_k into six mutually disjoint classes. Class I consists of those curves B_k belonging to the boundary of D where E contains a closed Jordan region bounded by B_k . Class II consists of the curves B_k each belonging to the boundary of D such that E contains a closed region bounded wholly by B_k plus one or more other B_j not belonging to the boundary of D . Class III consists of the curves B_k belonging to the boundary of D where the points of B_k are not limit points of interior points of E , where D lies on precisely one bank of B_k and neither D nor E on the other bank; here the hypothesis of Theorem 1 requires $f(z)$ to be merely continuous (not analytic) on B_k . Class IV consists of the curves B_k belonging to the boundary of D where D lies on both banks of B_k ; here $f(z)$ is analytic on B_k . Class V consists of those curves B_k each belonging to the boundary of D and to the boundary of a multiply connected subregion of E , a subregion bounded in part by some other B_j abutting on D . Class VI consists of the curves B_k not belonging to the boundary of D ; each such B_k is part of the boundary of a multiply connected subregion of E bounded by a subset of B . Each curve of Class III or VI bounds a Jordan region containing no point of $D+E$, by the connectedness of $D+E$, and conversely if B_k bounds a Jordan region containing no point of $D+E$ it is of Class III or VI. It will not be necessary for us to use integrals extended over every B_k in order to represent $f(z)$ in the interior points of $E+D_\rho$, but merely over those B_k of Classes III and VI, for the other B_k lie in the interior of $E+D_\rho$, in which interior $f(z)$ is analytic.

An integral of the form in (4), where B_k is of Class I, represents the

function identically zero for z in D_ρ near B_k ; indeed, B_k is homotopic to zero in a region of analyticity of the integrand. Such an integral as in (4), where B_k is of Class II, is equivalent to an integral taken over a set of other curves B_j each of Class VI, for z in D_ρ near B_k . If B_k is of Class III, the function represented as in (4) is analytic throughout $D+E$ except on B_k itself, and is by convention continuous on B_k . If B_k in (4) is of Class IV or V, the integral need not be considered in Cauchy's formula when z lies in $E+D_\rho$ for B_k consists wholly of interior points of $E+D_\rho$; for z in a neighborhood of B_k the function $f(z)$ is represented by Cauchy integrals taken over other parts of $B+\Gamma_\rho$. If B_k is of Class VI, the integral in (4) represents a component of $f(z)$ analytic throughout one of the two regions into which B_k separates the plane, a region containing all of $D+E$ except B_k itself; this component is continuous on B_k .

In recapitulation, for z in the interior points of $E+D_\rho$ the function $f(z)$ is represented by Cauchy's integral taken over the components of Γ_ρ as in (3) plus integrals as in (4) taken over all curves B_k of Classes III and VI. Each of these integrals as in (4) represents a component of $f(z)$, a function $\phi_2(z)$ analytic throughout one of the two Jordan regions into which B_k separates the plane, a region containing $D+E-B_k$. Each integral as in (3) represents a component $\phi_1(z)$ of $f(z)$ analytic throughout each of the two Jordan regions into which $\Gamma^{(i)}$ separates the plane, thus $\phi_1(z)$ is analytic throughout the extended plane except on $\Gamma^{(i)}$.

We now return to (6) and (7). Let $\Phi_0(z)$ be the sum of the functions $\phi_1(z)$ just enumerated and $\Phi_n(z)$ the sum for each n of the corresponding functions $f_{nj}(z)$; we obtain

$$(8) \quad |\Phi_0(z) - \Phi_n(z)| \leq A_5 e^{-n\rho} / n^{p+\alpha}, \quad z \text{ on } E,$$

$$(9) \quad |\Phi_n(z)| \leq A_6 e^{n(1-\rho)} / n^{p+\alpha}, \quad z \text{ in } D+E.$$

By the discussion already given we may now set for z on E and $D+E$ respectively

$$f(z) \equiv \Phi_0(z) + \sum \phi_2(z), \quad f_n(z) \equiv \Phi_n(z) + \sum \phi_2(z),$$

where the $\phi_2(z)$ are defined by integrals over curves of Classes III and VI only, whence (1) and (2) follow from (8) and (9).

We mention explicitly the fact that a Jordan curve $\Gamma^{(i)}$ may bound a Jordan region $J^{(i)}$ containing d_ρ , where $J^{(i)}$ contains also other Jordan curves $\Gamma^{(k)}$ belonging to Γ_ρ which bound Jordan regions $J^{(k)}$ containing the original d_ρ . However, on the subset E' of E in $J^{(i)}$ the function $f(z)$ is represented by the Cauchy integral of $f(z)$ taken over

all the curves $\Gamma^{(k)}$ in the closure of $J^{(i)}$ (namely functions $\phi_1(z)$) plus the Cauchy integral of $f(z)$ over the curves B_k of Classes III and VI in $J^{(i)}$ (namely functions $\phi_2(z)$). On any subset E'' of E exterior to $J^{(i)}$ the sum of the functions of $\phi_1(z)$ and $\phi_2(z)$ pertaining to the curves $\Gamma^{(k)}$ and B_k in the closure of $J^{(i)}$ is zero, for it is the sum of the Cauchy integrals of $f(z)$ over the complete boundary of one or more regions in which $f(z)$ is analytic, in the closures of which $f(z)$ is continuous. For z on E the function $f(z)$ is the sum of all functions $\phi_1(z)$ and of all functions $\phi_2(z)$ defined by integrals over B_k of Classes III and VI.

The present Theorem 1 includes [1, Theorem 6] and the first parts of [1, Theorems 1 and 3]. Moreover, we have expressed (1) and (2) in terms of the sequence $n = 1, 2, \dots$, but they can also be expressed in terms of a continuous parameter λ , as in §8 of [1].

In the direction of a converse to the present Theorem 1, the second part of [1] Theorem 1 is applicable.

REFERENCE

1. J. L. Walsh, *On approximation by bounded analytic functions*, Trans. Amer. Math. Soc. vol. 87 (1958) pp. 467-484.

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