NOTE ON A THEOREM OF FUGLEDE AND PUTNAM

S. K. BERBERIAN

1. An *involution* in a ring A is a mapping $a \rightarrow a^*$ ($a \in A$) such that $a^{**} = a$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$. An element $a \in A$ is (1) *normal* if $a^*a = aa^*$, (2) *self-adjoint* if $a^* = a$, (3) *unitary* if $a^*a = aa^* = 1$ ($1 =$ unity element of A). We say that "Fuglede's theorem holds in A " in case the relations $a \in A$, a normal, $b \in A$, $ba = ab$, imply $ba^* = a^*b$; briefly, A is an *FT-ring*.

It follows from a theorem of B. Fuglede that the ring A of all bounded operators in a Hilbert space (hence any adjoint-containing subring thereof) is an FT-ring [3, Theorem I]. For this ring, C. R. Putnam obtained the following generalization [9, Lemma]: if a_1, a_2 are normal, and $ba_1 = a_2b$, then $ba_1^* = a_2^*b$. A ring with involution, in which the latter theorem holds, will be called a *PT-ring*.

We denote by A_n the ring of all $n \times n$ matrices $x = (a_{ij})$, $a_{ij} \in A$, provided with the "conjugate-transpose" involution $x^* = (a_{ji}^*)$.

THEOREM 1. *If A_2 is an FT-ring, then A is a PT-ring.*

PROOF. Suppose a_1, a_2 are normal elements of A , and $ba_1 = a_2b$. Define

$$x = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

Clearly x is normal. Moreover,

$$yx = \begin{pmatrix} 0 & 0 \\ ba_1 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 0 \\ a_2b & 0 \end{pmatrix}$$

thus $yx = xy$. Since Fuglede's theorem holds in A_2 , $yx^* = x^*y$, in other words $ba_1^* = a_2^*b$.

EXAMPLE 1. Let A be an involutive (i.e. adjoint-containing) ring of bounded operators acting on a Hilbert space H . Then A_2 is an involutive ring acting on the direct sum of two copies of H . By Fuglede's theorem, A_2 is an FT-ring; thus A is a PT-ring by Theorem 1. This is Putnam's generalization of the Fuglede theorem [9, Lemma]. The argument extends easily to cover the case that a_1, a_2 are possibly unbounded. The result then reads: if $ba_1 \subset a_2b$ then $ba_1^* \subset a_2^*b$.

In the reverse direction, if A is a PT-ring, then Fuglede's theorem

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holds for the diagonal normal elements of A_2 ; we omit the obvious proof:

THEOREM 2. *If A is a PT-ring, and $a_1, a_2 \in A$ are normal, then the commutant of the normal matrix*

$$x = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

in A_2 is involutive; that is, the relations $y \in A_2, yx = xy$, imply $yx^ = x^*y$.*

A ring A with involution is said to satisfy the *square root axiom* [6, Chapter VII] in case: given any $a \in A$, there exists a self-adjoint element r such that $r^2 = a^*a$, and such that r is in the double commutant of a^*a (that is, the relation $b(a^*a) = (a^*a)b$ implies $br = rb$). **EXAMPLES:** any C^* -algebra (see [7, Theorem 26A]); the regular ring of a finite AW*-algebra [1, Corollary 6.2]. Suppose A is a ring satisfying the SR-axiom, and $a \in A$ is invertible. Write $u = ar^{-1}$, where r is the self-adjoint described above; clearly $u^*u = uu^* = 1$. The factorization $a = ur$ is called a "polar decomposition" for a .

THEOREM 3. *Let A be a PT-ring satisfying the square-root axiom. If a_1, a_2 are similar normal elements, they are unitarily equivalent.*

PROOF. Suppose $ba_1b^{-1} = a_2$. Then $ba_1 = a_2b$; since A is a PT-ring, $ba_1^* = a_2^*b$, thus $a_1b^* = b^*a_2$. Let $b = ur$ be a polar decomposition. Then a_1 commutes with b^*b ; for, $a_1(b^*b) = (a_1b^*)b = (b^*a_2)b = b^*(a_2b) = b^*(ba_1) = (b^*b)a_1$. Hence $a_1r = ra_1$, and $a_2 = ba_1b^{-1} = (ur)a_1(r^{-1}u^*) = ua_1rr^{-1}u^* = ua_1u^*$.

EXAMPLE 2 (PUTNAM). If A is the ring of all bounded operators in a Hilbert space, and $a_1, a_2 \in A$ are similar normal operators, then a_1, a_2 are unitarily equivalent by Example 1 and Theorem 3 (see [9, Theorem I]). The argument works just as well for A any C^* -algebra, the point being that the elements implementing the similarity and unitary equivalence are to be drawn from A .

A ring A with involution is said to possess a *trace* if there exists a mapping $a \rightarrow \text{tr}(a)$ of A into some abelian group, such that (1) $\text{tr}(a+b) = \text{tr}(a) + \text{tr}(b)$, (2) $\text{tr}(ab) = \text{tr}(ba)$, and (3) $\sum_1^k \text{tr}(a_i^*a_i) = 0$ implies $a_1 = \dots = a_k = 0$.

THEOREM 4. *If A is a ring with involution and trace, then A is a PT-ring.*

PROOF. Since A_n also has a trace, defined for a matrix $x = (a_{ij})$ by the formula $\text{tr}(x) = \sum_1^n \text{tr}(a_{ii})$, it will suffice by Theorem 1 to show that A is an FT-ring. Suppose x is normal, and $yx = xy$. It must be

shown that $z = yx^* - x^*y$ is 0. We learned the ensuing argument for this from I. Kaplansky. One has

$$\begin{aligned} zz^* &= yx^*xy^* - yx^*y^*x - x^*xyy^* + x^*yy^*x \\ &= yxx^*y^* - yx^*y^*x - x^*xyy^* + x^*yy^*x \\ &= xyx^*y^* - yx^*y^*x - xx^*yy^* + x^*yy^*x. \end{aligned}$$

Since $\text{tr}(xyx^*y^*) = \text{tr}(yx^*y^*x)$, and $\text{tr}(xx^*yy^*) = \text{tr}(x^*yy^*x)$, one has $\text{tr}(zz^*) = 0$, hence $z = 0$.

EXAMPLE 3. Let A be a commutative ring with involution, such that $\sum_1^k a_i^* a_i = 0$ implies $a_1 = \cdots = a_k = 0$, and set $\text{tr}(a) = a$. Then A_n is a PT-ring by Theorem 4.

EXAMPLE 4. Let Q be the ring of all real quaternions $a = \alpha + \beta i + \gamma j + \delta k$, with involution $a^* = \alpha - \beta i - \gamma j - \delta k$. One has $a^* a = a a^* = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$, so that incidentally every element of Q is normal. Set $\text{tr}(a) = \alpha$. It results from Theorem 4 that Q_n is a PT-ring. This is Putnam's theorem for finite-dimensional quaternionic Hilbert space, and raises the analogous question for infinite dimension.

EXAMPLE 5. Let A be a homogeneous AW*-algebra of finite order n , so that $A = Z_n$, where Z is the center of A . Let C be the regular ring of A , W the regular ring of Z ; we may identify W with the center of C [1, Theorem 9.2]. Now, W has the properties in Example 3 [1, Lemma 3.4]; since $C = W_n$ [2, concluding remark (2)], it follows that C has a W -valued trace. Thus C is a PT-ring. See Theorem 5 for the generalization to A of finite Type I.

LEMMA. Suppose A is the C^* -sum of a family (A_i) of finite AW*-algebras, C is the regular ring of A , and C_i is the regular ring of A_i . Then C is the complete direct sum of the C_i .

PROOF. According to [5, §2], A is the set of all families $a = (a_i)$ with $a_i \in A$ and $\|a_i\|$ bounded; the operations in A are coordinatewise. One knows from [5] that A is an AW*-algebra, and is clearly of finite class, so that we may speak of its regular ring C .

Let D be the complete direct sum of the C_i . That is, D is the set of all families $x = (x_i)$ with $x_i \in C_i$, with the coordinatewise operations. By an easy coordinatewise argument, one sees that D is regular. It must be shown that $D = C$.

We may identify A as an involutive subalgebra of D . We shall prove $D = C$ by verifying the criterion of [1, §11]. Suppose $x, y, z \in D$, and $x^*x + y^*y + z^*z = 1$. Then $x_i^*x_i + y_i^*y_i + z_i^*z_i = 1$ for all i , hence $x_i, y_i, z_i \in A_i$; since these elements all have norm ≤ 1 , one has $x, y, z \in A$.

THEOREM 5. *If A is a finite AW^* -algebra of Type I, its regular ring C possesses a center-valued trace. In particular, C is a PT-ring.*

PROOF. Write A as the C^* -sum of a family (A_i) of homogeneous algebras, and let C_i be the regular ring of A_i . By the Lemma, C is the complete direct sum of the C_i . It follows at once that the center W of C is the complete direct sum of the centers W_i of C_i . According to Example 5, C_i has a W_i -valued trace. Then $(x_i) \rightarrow (\text{tr } x_i)$ defines a W -valued trace on C , thus C is a PT-ring by Theorem 4.

It is reasonable to suppose that C is a PT-ring, for any finite AW^* -algebra A ; in any case, since A_2 is AW^* with regular ring C_2 by [2], it would suffice by Theorem 1 to show that C is an FT-ring.

COROLLARY. *A , C as in Theorem 5. If z_1, z_2 are similar normal elements of C , they are unitarily equivalent.*

PROOF. C is a PT-ring, with square root axiom [1, Corollary 6.2]; quote Theorem 3.

It results from the corollary that if two normal elements are similar via an unbounded element, they are already similar via a bounded (even unitary) element; in particular, a normal bounded element cannot be similar to a normal unbounded element. Normality is essential here, as is shown by the following example due to Jacob Feldman:

EXAMPLE 6. Let A be the C^* -sum of denumerably many copies of the algebra K_2 of 2×2 complex matrices. A may be represented as the algebra of all functions $n \rightarrow f(n)$ ($n = 1, 2, 3, \dots$), with $f(n) \in K_2$, $\|f(n)\|$ bounded, and operations pointwise. Since K_2 is its own regular ring, the regular ring C of A is the algebra of all functions $n \rightarrow f(n)$ with $f(n) \in K_2$. Consider the functions $f, g, h \in C$ defined by

$$f(n) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g(n) = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}, \quad h(n) = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}.$$

Since $h(n)f(n)h(n)^{-1} = g(n)$ for all n , one has $hfh^{-1} = g$. Thus f and g are similar in C , even though f is bounded (i.e., is an element of A) and g is not bounded.

2. More on the regular ring. Throughout, C denotes the regular ring of a finite AW^* -algebra A (of unrestricted type).

If $x \in C$, and $RP(x) = 1$, then x is invertible. For, $Cx = C$ [1, Corollary 7.1], so there exists $y \in C$ with $yx = 1$; moreover $LP(x) \sim RP(x) = 1$, hence $LP(x) = 1$ by finiteness, $xC = C$, $xz = 1$ for suitable z . Note that an $x \in C$ is invertible if and only if it is left (right) invertible.

If $x \in C$ is invertible, then x^* is invertible, and $(x^*)^{-1} = (x^{-1})^*$; if moreover x is self-adjoint, so is x^{-1} .

LEMMA 1. *If $x \in C$, $x \geq 0$, and x is invertible, then $x^{-1} \geq 0$.*

PROOF. Say $xy = yx = 1$, and $x = z^*z$ [1, Definition 6.1]. Then $(yz^*)z = 1$ shows that z is invertible (see above remarks), hence $x^{-1} = (z^*z)^{-1} = (z^{-1})(z^{-1})^* \geq 0$.

LEMMA 2. *Let $a \in A$, $0 \leq a \leq 1$, and suppose a has an inverse in C . Then $a^{-1} \geq 1$.*

PROOF. Say $ax = xa = 1$; we know $x \geq 0$ from Lemma 1. Write $x = y^2$, y self-adjoint [1, Corollary 6.2]. Since $(ay)y = y(ya) = 1$, y is invertible, and $ay = ya = y^{-1}$. Then $a \leq 1$, $y^*ay \leq y^*y$, $yay \leq y^2$, $ay^2 \leq y^2$, $ax \leq x$, $1 \leq x$.

THEOREM 6. *Suppose $x, y \in C$, $0 \leq x \leq y$, and x is invertible. Then y is invertible, and $x^{-1} \geq y^{-1} \geq 0$.*

PROOF. The relation $0 \leq x \leq y$ implies $RP(x) \leq RP(y)$ (see the proof of [1, Corollary 7.6]); by assumption $RP(x) = 1$, hence $RP(y) = 1$, y is invertible. Write $x^{1/2} = wy^{1/2}$ with $w \in A$, $w^*w \leq 1$ [1, Corollary 7.6]. Then $w = (x^{1/2})(y^{1/2})^{-1}$ is invertible in C , hence so is w^*w , and $(w^*w)^{-1} = (w^{-1})(w^{-1})^* \geq 1$ by Lemma 2. Since $(x^{1/2})^{-1} = (y^{1/2})^{-1}w^{-1}$, one has $x^{-1} = (x^{1/2})^{-1}(x^{1/2})^{-1*} = (y^{1/2})^{-1}(w^{-1})(w^{-1})^*(y^{1/2})^{-1} \geq (y^{1/2})^{-1} \cdot (y^{1/2})^{-1} = y^{-1}$. For a similar result of Rellich, see [4, Hilfsatz 4].

COROLLARY. *Suppose A has the property that every increasingly directed family of self-adjoint elements, which is bounded above, has a least upper bound. Then C has the same property.*

PROOF. For ease of notation, we write the proof for sequences. Suppose $x_i \in C$ are self-adjoint, $x_1 \leq x_2 \leq x_3 \leq \dots$, $y \in C$ is self-adjoint, and $x_i \leq y$ for all i . Adding $-x_1$ throughout, we can assume $0 \leq x_i \leq y$. Then $1 \leq 1 + x_1 \leq 1 + x_2 \leq \dots \leq 1 + y$, hence by Theorem 6, $1 \geq (1 + x_1)^{-1} \geq (1 + x_2)^{-1} \geq \dots \geq (1 + y)^{-1} \geq 0$. But $(1 + x_i)^{-1}$ and $(1 + y)^{-1}$ belong to A [1, Lemma 5.1]. Let $a \in A$ be the greatest lower bound of the $(1 + x_i)^{-1}$; one has $0 \leq (1 + y)^{-1} \leq a \leq (1 + x_i)^{-1}$. By Theorem 6, a has an inverse in C , and $1 + y \geq a^{-1} \geq 1 + x_i$. Evidently $a^{-1} - 1$ is a least upper bound for the x_i . (EXAMPLE: A any finite W^* -algebra; see [8, Theorem 1].)

LEMMA. *Let $z \in C$ be normal, and suppose there exists a complex number λ such that $z - \lambda$ has an inverse in A . Then the relations $a \in A$, $az = za$, imply $az^* = z^*a$.*

PROOF. (We are assuming, so to speak, that the "resolvent set" of z is nonempty.) Suppose $a \in A$, $az = za$. Then $a(z - \lambda) = (z - \lambda)a$, and $z - \lambda$ is normal. Changing notation, assume z invertible, $z^{-1} \in A$,

$az = za$. Then $z^{-1}a = az^{-1}$, hence by Fuglede's theorem $a(z^{-1})^* = (z^{-1})^*a$, $a(z^*)^{-1} = (z^*)^{-1}a$, $z^*a = az^*$.

THEOREM 7. *Let $z \in C$ be normal, and write $z = x + iy$ with x and y self-adjoint. Suppose there exists a real number α such that $x - \alpha$ (or $y - \alpha$) has an inverse in A . Then the relations $a \in A$, $az = za$, imply $az^* = z^*a$.*

PROOF. Passing to iz if necessary, we may suppose that it is $x - \alpha$ which has a bounded inverse. Then $(x - \alpha)^{-2} = (x - \alpha)^{-1*}(x - \alpha)^{-1} \leq \beta$ for a suitable real number $\beta > 0$. By Theorem 6, $(x - \alpha)^2 \geq 1/\beta > 0$. Since $yx = xy$ by normality, and $z - \alpha = (x - \alpha) + iy$, we have $(z - \alpha)^* \cdot (z - \alpha) = (x - \alpha)^2 + y^2 \geq (x - \alpha)^2 \geq 1/\beta > 0$. Hence $(z - \alpha)^*(z - \alpha)$ is invertible, and $(z - \alpha)^{-1}(z - \alpha)^{-1*} \leq \beta$. Therefore $(z - \alpha)^{-1} \in A$ [1, Lemma 5.1]; quote the lemma.

A self-adjoint $x \in C$ is *semi-bounded* in case there exists a real number β such that either $x \leq \beta$ or $x \geq \beta$. For instance if $x \in A$ is self-adjoint, then $x \leq \|x\|$. If x is semi-bounded, say $x \geq \beta$, then setting $\alpha = \beta - 1$, one has $x - \alpha \geq 1$, hence $x - \alpha$ has a bounded inverse (Theorem 6, and Lemma 5.1 of [1]). Thus:

COROLLARY. *Let $z \in C$ be normal, and write $z = x + iy$, with x and y self-adjoint. Suppose either x or y is semi-bounded. Then the relations $a \in A$, $az = za$, imply $az^* = z^*a$.*

If A has a trace (e.g. if A is Type I, or is a finite W^* -algebra), it is clear that the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. We do not know if every finite AW^* -algebra A has this property, but whenever A does, so does C :

THEOREM 8. *Suppose the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. Then the relations $x \in C$, $x^*x \leq xx^*$, imply $x^*x = xx^*$.*

PROOF. Suppose $x^*x \leq xx^*$. Write $x = ur$, $r \geq 0$, u unitary [1, Corollary 7.4]. Then $x^*x = r^2$, and $xx^* = ur^2u^* = u(x^*x)u^*$. Setting $s = x^*x$, $t = xx^*$, we have $0 \leq s \leq t$, and s , t are unitarily equivalent. Set $b = (1 + s)^{-1}$, $c = (1 + t)^{-1}$; clearly b , c are unitarily equivalent, in fact $usu^* = t$ yields $ubu^* = c$. Moreover $b \geq c$ by Theorem 6, and $b, c \in A$ [1, Lemma 5.1]. Set $a = b^{1/2}u^*$. Then $aa^* = b^{1/2}u^*ub^{1/2} = b \geq c = ubu^* = a^*a$. By the hypothesis on A , $aa^* = a^*a$, hence $b = c$, and this leads to $s = t$.

COROLLARY 1. *Suppose the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. If $x \in C$, and x commutes with x^*x , then x is normal.*

PROOF. By assumption $xx^*x = x^*xx$. Right-multiplying by x^* ,

$xx^*xx^*=x^*xxx^*$. Setting $r=x^*x$, $s=xx^*$, we have $r \geq 0$, $s \geq 0$, and $s^2=rs$. In particular rs is self-adjoint, so that $rs=sr$. Hence by uniqueness of positive square roots, $s=(s^2)^{1/2}=(rs)^{1/2}=r^{1/2}s^{1/2}$ (see [1, remarks following Definition 6.3]). Then $0 \leq (r^{1/2} - s^{1/2})^2 = r - 2r^{1/2}s^{1/2} + s = r - 2s + s = r - s$, thus $0 \leq s \leq r$. That is, $xx^* \leq x^*x$, hence $xx^*=x^*x$ by Theorem 8.

REMARK. In an *infinite* algebra B , choose $x \in B$ with $x^*x=1$ but $xx^* \neq 1$. Then x commutes with x^*x , but is not normal.

COROLLARY 2. Suppose the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. Then every triangular normal matrix in C_n is diagonal.

PROOF. Suppose e.g. $n=3$, and

$$z = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

is a normal element of C_3 . From the 1-1 position in the relation $z^*z=zz^*$, one has $a^*a=aa^*+bb^*+cc^*$, thus $a^*a-aa^*=bb^*+cc^* \geq 0$ [1, Theorem 6.1], $aa^* \leq a^*a$. By Theorem 8, $aa^*=a^*a$, hence $b=c=0$ [1, Lemma 3.4]. Inspection now of the 2-2 position similarly yields $e=0$. The case for general n is an obvious induction.

REMARK. If B is an *infinite* algebra, there exists a normal (even unitary) matrix

$$x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

in B_2 with $b \neq 0$. For, choose a partial isometry $a \in A$ with $a^*a=1$, $aa^*=e \neq 1$, and set $b=1-e$, $c=a^*$.

Addenda. (1) I am indebted to J. Dixmier for calling my attention to the references [4] and [8].

(2) Recently M. Rosenblum has given a beautiful proof of the Fuglede-Putnam theorem; for bounded operators, the proof is non-spatial (see [10]).

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STATE UNIVERSITY OF IOWA

A CORRECTION AND IMPROVEMENT OF A THEOREM ON ORDERED GROUPS

PAUL CONRAD

In this note the notation and terminology of [1] will be used throughout. In particular, G will always denote an o -group with well ordered rank. Let P be the multiplicative group of positive rational numbers, and let R be the additive group of real numbers. In [1] the proofs of Theorems 2 and 3 are incorrect. This is a result of the careless formulation of the theorems by the author. Consider the following properties of G .

(1) Each component G_γ/G_γ of G has its group of o -automorphisms isomorphic to a subgroup P_γ of P .

(2) Each component G_γ/G_γ of G is o -isomorphic to a subgroup D_γ of R , and the only o -automorphisms of D_γ are multiplications by some elements of P .

(3) For each pair $\alpha \in \mathfrak{A}$ and $\gamma \in \Gamma$, there exists a pair m, n of positive integers such that $n\alpha \equiv mg \pmod{G_\gamma}$ for all g in G_γ .

The statements of Theorems 2 and 3 include the hypothesis (1), but (2) and (3) are actually used in the proofs. Clearly (2) implies (1).

LEMMA. (a) (2) is independent of the particular choice of D_γ . (b) (2) implies (3). (c) (1) does not imply (2) or (3).

PROOF. (a) Let σ be an o -isomorphism of the subgroup A of R onto the subgroup B of R , and suppose that the only o -automorphisms of A are multiplications by some elements of P . If β is an o -automor-