A PROBLEM IN LINEAR DIFFERENTIAL EQUATIONS¹

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In the Bulletin of the American Mathematical Society, March 1958, page 61, R. E. Bellman suggests the following for research.

"Consider the second order linear differential equation

$$(1) \quad u'' + (1 + \lambda g(x))u = 0$$

where λ is a real constant and $\int_0^\infty |g(x)| dx < \infty$. Let $u_1(x)$ be the solution specified by $u_1(0) = 0$, $u_1'(0) = 1$. It is known that

$$u_1(x) \sim r(\lambda) \sin(x + \theta(\lambda)).$$

Taking λ to be a complex variable, what are the analytic properties of the functions $r(\lambda)$ and $\theta(\lambda)$? In particular, where are the singularities nearest the origin? If $g(x) > 0$ for $x \geq 0$, is the singularity nearest the origin on the negative axis?" We assume that $g(x)$ is real for real x .

1. It is known² that there are two solutions of (1) which we denote by $u_3(x)$ and $u_4(x)$ such that when $x \rightarrow \infty$

$$(2) \quad \begin{aligned} u_3(x) - \sin x &\rightarrow 0, \\ u_3'(x) - \cos x &\rightarrow 0, \\ u_4(x) - \cos x &\rightarrow 0, \\ u_4'(x) + \sin x &\rightarrow 0. \end{aligned}$$

This approach is uniform in λ so long as $|\lambda| < M$. This last statement does not seem to have been made in the literature. However, it is readily inferred from text book proofs.

2. Let $u_1(x)$ and $u_2(x)$ be solutions of (1) such that

$$(3) \quad \begin{aligned} u_1(0) &= 0, & u_1'(0) &= 1, \\ u_2(0) &= 1, & u_2'(0) &= 0. \end{aligned}$$

Choose $b > 0$. Let $U_1(x)$ and $U_2(x)$ be solutions of (1) such that

$$(4) \quad \begin{aligned} U_1(b) &= \sin b, & U_1'(b) &= \cos b, \\ U_2(b) &= \cos b, & U_2'(b) &= -\sin b. \end{aligned}$$

The two functions $u_1(x)$ and $u_2(x)$ form a fundamental system of solu-

Presented to the Society, November 29, 1958; received by the editors August 22, 1958.

¹ This research was supported by the United States Air Force.

² Coddington and Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955, p. 92.

tions of (1). Similarly $u_3(x)$ and $u_4(x)$ form a fundamental system and likewise $U_1(x)$ and $U_2(x)$.

Since $u_3(x)$, $u'_3(x)$, $u_4(x)$, $u'_4(x)$ are bounded, all solutions of (1) and their first and second derivatives are bounded as is already well known. We write

$$(5) \quad \begin{aligned} u_1(x) &= c_{11}^{(b)}(\lambda)U_1(x) + c_{12}^{(b)}(\lambda)U_2(x), \\ u_2(x) &= c_{21}^{(b)}(\lambda)U_1(x) + c_{22}^{(b)}(\lambda)U_2(x) \end{aligned}$$

and

$$(6) \quad \begin{aligned} u_1(x) &= e_{11}(\lambda)u_3(x) + e_{12}(\lambda)u_4(x), \\ u_2(x) &= e_{21}(\lambda)u_3(x) + e_{22}(\lambda)u_4(x). \end{aligned}$$

We note that the c 's are independent of x .

We shall prove the following theorem:

THEOREM 1. *The functions $c_{ij}(\lambda)$, $i, j=1, 2$ can be analytically continued for complex λ so that each is an analytic function of λ so long as $|\lambda| \leq M$, where M is arbitrary.*

We form the difference equation in discrete argument

$$(7) \quad \begin{aligned} (n/b)^2 \Delta^2 y(x_i) + [1 + \lambda g(x_i)]y(x_{i+1}) &= 0, \\ x_0 = 0, \quad x_{i+1} = x_i + b/n, \quad \Delta y(x_i) &= y(x_{i+1}) - y(x_i). \end{aligned}$$

Here i and n are integers. Let $y_1(x_i)$ and $y_2(x_i)$ be those solutions of (7) such that

$$(8) \quad \begin{aligned} y_1(0) &= 0, \quad n/b \Delta y_1(0) = 1, \\ y_2(0) &= 1, \quad \Delta y_2(0) = 0. \end{aligned}$$

Also let $Y_1(x_i)$ and $Y_2(x_i)$ be solutions of (7) such that

$$(9) \quad \begin{aligned} Y_1(b) &= \sin b, \quad \Delta Y_1(b) = \Delta \sin b, \\ Y_2(b) &= \cos b, \quad \Delta Y_2(b) = \Delta \cos b. \end{aligned}$$

Now count backward from b . By the use of (7) we find that $Y_1(0)$, $\Delta Y_1(0)$, $Y_2(0)$, $\Delta Y_2(0)$ are all polynomials in λ . We now write

$$(10) \quad \begin{aligned} y_1(x_i) &= k_{11}^{(b)}(\lambda)Y_1(x_i) + k_{12}^{(b)}(\lambda)Y_2(x_i), \\ \Delta y_1(x_i) &= k_{11}^{(b)}(\lambda)\Delta Y_1(x_i) + k_{12}^{(b)}(\lambda)\Delta Y_2(x_i) \end{aligned}$$

and

$$(11) \quad \begin{aligned} y_2(x_i) &= k_{21}^{(b)}(\lambda)Y_1(x_i) + k_{22}^{(b)}(\lambda)Y_2(x_i), \\ \Delta y_2(x_i) &= k_{21}^{(b)}(\lambda)\Delta Y_1(x_i) + k_{22}^{(b)}(\lambda)\Delta Y_2(x_i). \end{aligned}$$

The k 's are independent of x_i . Choose $x_i=0$. Note that $Y_1(x_i)\Delta Y_2(x_i) - Y_2(x_i)\Delta Y_1(x_i)$ is a constant and can be evaluated at b . Solve for the k 's. We find

$$(12) \quad k_{11}^{(b)}(\lambda) = (b/n \csc b/n) Y_2(0)$$

with similar forms for the other k 's. These forms show that the k 's are polynomials in λ . If we solve (5) for the c 's with $x=0$ we find

$$(13) \quad c_{11}^{(b)}(\lambda) = U_2(0).$$

We make all solutions of (7) functions of the continuous variable x by connecting $y(x_i)$ and $y(x_{i+1})$ when plotted in the Cartesian plane by straight line segments. We write $y(x)$.

Now let $n \rightarrow \infty$ holding b fixed. As is well known the y 's approach the corresponding u 's uniformly in x and λ when $0 \leq x \leq b$ and $|\lambda| \leq M$. Hence $Y_2(0) \rightarrow U_2(0)$ uniformly in λ . Hence $k_{ij}^{(b)}(\lambda) \rightarrow c_{ij}^{(b)}(\lambda)$, $i, j=1, 2$ uniformly in λ . Hence the $c_{ij}^{(b)}(\lambda)$ are analytic in λ when $|\lambda| \leq M$.

3. We now let $b \rightarrow \infty$. As already noted the c 's are independent of x and can be determined at any point. With the aid of (4) and (5) we find

$$c_{11}^{(b)}(\lambda) = u_1(b) \sin b + u_1'(b) \cos b.$$

Let $u_3(x) = \sin x + \epsilon(x)$, $u_4(x) = \cos x + \eta(x)$ then from (6)

$$c_{11}(\lambda) = \frac{u_1(b)[\sin b - \eta'(b)] + u_1'(b)[\cos b + \eta(b)]}{[\sin b + \epsilon(b)][\sin b - \eta'(b)] + [\cos b + \eta(b)][\cos b + \epsilon'(b)]}.$$

Now $\epsilon(b)$, $\eta(b)$, $\epsilon'(b)$, $\eta'(b)$ approach zero uniformly in λ and $u(b)$ and $u'(b)$ are bounded. Hence $(c_{11}(\lambda) \rightarrow c_{11}^{(b)}(\lambda)) \rightarrow 0$ uniformly in λ . Since $c_{11}^{(b)}(\lambda)$ is analytic, $|\lambda| < M$, so is $c_{11}(\lambda)$. The same reasoning applies to $c_{12}(\lambda)$, $c_{21}(\lambda)$ and $c_{22}(\lambda)$.

This completes the proof of Theorem 1.

4. We know

$$\begin{aligned} u_1(x) &= c_{11}(\lambda)u_3(x) + c_{12}(\lambda)u_4(x) \sim c_{11}(\lambda) \sin x + c_{12}(\lambda) \cos x \\ &= r(\lambda) \sin(x + \theta(\lambda)), \quad |\lambda| < M \end{aligned}$$

where

$$\begin{aligned} [r(\lambda)]^2 &= [c_{11}(\lambda)]^2 + [c_{12}(\lambda)]^2, \\ \cos \theta(\lambda) &= c_{11}(\lambda)/r(\lambda). \end{aligned}$$

so long as λ is real. If we analytically continue the functions $[r(\lambda)]^2$

and $\cos \theta(\lambda)$ allowing λ to be complex we observe that $[r(\lambda)]^2$ is entire and that the only possible singularities of $\cos \theta(\lambda)$ are where $r(\lambda) = 0$. Now the Wronskian of $u_3(x)$ and $u_4(x)$ is a constant. It approaches -1 and consequently is -1 . With this in mind with the use of (3) and (6) we find $c_{11}(\lambda) = u_4(0)$, $c_{12}(\lambda) = -u_3(0)$. Hence $c_{11}(\lambda)$ and $c_{12}(\lambda)$ are not both zero else $u_3(x)$ and $u_4(x)$ would have a Wronskian equal to zero which they do not. We conclude among other things that if $r(\lambda) = 0$ then $c_{11}(\lambda) \neq 0$, $c_{12}(\lambda) \neq 0$.

Let λ be real. Then $c_{11}(\lambda)$ and $c_{12}(\lambda)$ are real and $r(\lambda)$ can not vanish unless both vanish simultaneously. Hence the zeros of $r(\lambda)$ are all imaginary. We note in passing that $r(0) = 1$.

We have proved the following theorem:

THEOREM 2. *If $[r(\lambda)]^2$ and $\cos \theta(\lambda)$ are analytically continued in the complex plane, then $[r(\lambda)]^2$ is an entire function and $\cos \theta(\lambda)$ is in general analytic. Its only finite singularities are at the points where $r(\lambda) = 0$. Such points are necessarily imaginary.*

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