ON CONDITIONAL INCLUSION OF MATRIX METHODS

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A summation method B is said to be stronger than a method A if the set of A -summable sequences is contained in the set of B -summable sequences. The purpose of this paper is to study inclusion relations between methods for other sets of sequences. In the first section we shall introduce the notation. In the second we prove the two main theorems, and present some applications in the third part. Finally we shall mention possible generalizations and corresponding theorems for absolute summability.

1. A sequence $\{s_n\}$ is summable by the triangular matrix method¹ $A = (a_{n\nu})$, if the sequence

$$\sigma_n = \sum_{\nu=0}^n a_{n\nu} s_\nu$$

converges for $n \rightarrow \infty$. We denote by $A' = (a'_{n\nu})$ the inverse matrix of A , i.e. A' is the matrix such that

$$s_n = \sum_{\nu=0}^n a'_{n\nu} \sigma_\nu.$$

With $\Omega(n)$ denoting a positive monotonic function, defined for all natural numbers² we shall let $[A, O(\Omega)]$ stand for the set of all sequences $\{s_n\}$ for which $\sigma_n = O(\Omega(n))$, $n \rightarrow \infty$. The symbol $[A, o(\Omega)]$ is defined in a similar way. We say, the method $C = (c_{n\nu})$ satisfies the mean value condition ($\mathfrak{M}$), if to any pair of integers $n \leq m$ there exists a third integer n' such that

$$\left| \sum_{\nu=0}^n c_{m\nu} s_\nu \right| \leq K \left| \sum_{\nu=0}^{n'} c_{n'\nu} s_\nu \right|, \quad n' \leq n \leq m$$

where K is a constant depending only on the method C .³ It has been shown by Jurkat and Peyerimhoff [4], that many important methods satisfy the condition ($\mathfrak{M}$), e.g. the (C, k) methods for $0 \leq k \leq 1$, and the Riesz methods for $0 \leq k \leq 1$.⁴

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¹ We shall restrict our investigations to triangular matrices.

² $\omega(n)$ will be used with the same definition.

³ See Jurkat-Peyerimhoff [4; 5].

⁴ See also Bosanquet [1] and Jurkat [3].

2. THEOREM 1. *Let the method A satisfy the condition $(\mathfrak{M})$ and let $\Omega(n)$ be monotonic increasing. If*

$$S_n = \sum_{\nu=0}^n \Omega(\nu) \left| \Delta_\nu \frac{b_{n\nu}}{a_{n\nu}} \right| = O(1)$$

then

$$[A, O(\Omega)] \subset [B, O(1)].$$

PROOF. We have

$$\begin{aligned} \tau_n &= \sum_{\nu=0}^n b_{n\nu} s_\nu = \sum_{\nu=0}^n \frac{b_{n\nu}}{a_{n\nu}} a_{n\nu} s_\nu \\ &= \sum_{\nu=0}^n \Omega(\nu) \Delta_\nu \frac{b_{n\nu}}{a_{n\nu}} \frac{1}{\Omega(\nu)} \sum_{\mu=0}^\nu a_{n\mu} s_\mu. \end{aligned}$$

Therefore

$$\begin{aligned} |\tau_n| &\leq \sum_{\nu=0}^n \Omega(\nu) \left| \Delta_\nu \frac{b_{n\nu}}{a_{n\nu}} \right| \frac{1}{\Omega(\nu)} K \left| \sum_{\mu=0}^{n'} a_{n'\mu} s_\mu \right| \\ &= \sum_{\nu=0}^n \Omega(\nu) \left| \Delta_\nu \frac{b_{n\nu}}{a_{n\nu}} \right| \frac{\Omega(n')}{\Omega(\nu)} O(1), \quad n' \leq \nu \leq n. \end{aligned}$$

Since $\Omega(n)$ is monotonic increasing, $\Omega(n')/\Omega(\nu) = O(1)$ and hence the proof is complete.

REMARKS. (1) Apparently $S_n = O(\omega(n))$ implies $[A, O(\Omega)] \subset [B, O(\omega)]$. Also $S_n = o(\omega(n))$ implies $[A, O(\Omega)] \subset [B, o(\omega)]$.

(2) For $\Omega(n)=1$ Theorem 1 reduces to the inclusion theorem of Jurkat and Peyerimhoff [5, Satz 5].

THEOREM 2. *Let the method B be regular and let*

$$(1) \quad a'_{\mu\nu} \rightarrow 0 \quad \text{as} \quad \mu \rightarrow \infty.$$

Then

$$(2) \quad [A, O(\Omega)] \subset [B, o(1)]$$

if and only if

$$(3) \quad \sum_{\nu=0}^n \left| \Omega(\nu) \sum_{\mu=\nu}^n b_{n\mu} a'_{\mu\nu} \right| = o(1).^5$$

PROOF. We write again

$$\tau_n = \sum_{\mu=0}^n b_{n\mu} s_\mu,$$

⁵ In Theorem 2 $\Omega(n)$ may be monotonic increasing or decreasing.

then the inclusion (2) states that

$$\sigma_n = O(\Omega(n)) \quad \text{implies} \quad \tau_n = o(1).$$

Using the inverse transformation we obtain

$$\tau_n = \sum_{\mu=0}^n b_{n\mu} \sum_{\nu=0}^{\mu} a'_{\mu\nu} \sigma_{\nu} = \sum_{\nu=0}^n \sigma_{\nu} \sum_{\mu=\nu}^n b_{n\mu} a'_{\mu\nu}.$$

Now we introduce the sequence $\{\delta_{\nu}\}$, where $\sigma_{\nu} = \Omega(\nu)\delta_{\nu}$, so that $\delta_{\nu} = O(1)$. Therefore

$$\tau_n = \sum_{\nu=0}^n \delta_{\nu} \Omega(\nu) \sum_{\mu=\nu}^n b_{n\mu} a'_{\mu\nu} = \sum_{\nu=0}^n c_{n\nu} \delta_{\nu},$$

and $(c_{n\nu})$ is a matrix method which transforms every bounded sequence $\{\delta_{\nu}\}$ into a null sequence $\{\tau_n\}$. Condition (1) implies that $c_{n\nu} \rightarrow 0$ as $n \rightarrow \infty$; therefore we can apply the theorem by Schur⁶ to the effect that under these circumstances

$$\sum_{\nu=0}^n |c_{n\nu}| = o(1) \quad \text{as } n \rightarrow \infty.$$

With this the first part of the theorem is proved. To show that equation (3) is sufficient for the inclusion (2) presents no difficulty.

REMARKS. (3) This theorem allows extensions similar to those mentioned in Remark 1.

(4) If the matrix $(a_{n\nu})$ has the form

$$\begin{aligned} a_{n\nu} &= 1 & \text{for } \nu \leq n, \\ a_{n\nu} &= 0 & \text{for } \nu > n \end{aligned}$$

then $\Omega(n)$ is the "summability function of the second kind for the method B " introduced by Lorentz [6].

THEOREM 3. *Under the conditions of Theorem 2*

$$[A, O(\Omega)] \subset [B, O(1)] \quad ([A, o(\Omega)] \subset [B, o(1)])$$

if and only if

$$(4) \quad \sum_{\nu=0}^n \left| \Omega(\nu) \sum_{\mu=\nu}^n b_{n\mu} a'_{\mu\nu} \right| = O(1).$$

The proof is similar to that of Theorem 2.

3. Of the possible applications of these theorems we shall only

⁶ See Schur [7].

mention a few. By taking the method B in Theorem 2 to be the convergence matrix we obtain the

COROLLARY 1. *Let $a'_{nv} \rightarrow 0$ as $n \rightarrow \infty$, then*

$$\sigma_n^A = O(\Omega(n)) \quad (o(\Omega(n)))$$

is a Tauberian condition for the method A if and only if

$$\sum_{\nu=0}^n |\Omega(\nu) a'_{n\nu}| = o(1) \quad (O(1)).$$

As a consequence of Theorem 3 we find as a known result.⁷

COROLLARY 2. *A regular method A is equivalent to convergence if and only if*

$$\sum_{\nu=0}^n |a'_{n\nu}| = O(1).$$

If, in Theorem 3 we take A to be the method (C, k') and B to be the method (C, k) where $k < k'$, we find the

COROLLARY 3. *Let the sequence $\{s_n\}$ be (C, k') -summable and let*

$$\sigma_n^{k'} = O(n^{k-k'})$$

then the sequence is (C, k) -summable.

Theorem 3 also furnishes a simple proof for the following well-known theorem.

THEOREM 4. *Let the sequence $\{s_n\}$ be $(C, 1)$ -summable and let*

$$\sigma_n = s + o(n^{-1/2}),$$

then the sequence is Euler-summable.⁸

PROOF. We take A to be the $(C, 1)$ -matrix and B to be the Euler matrix. Then

$$b_{n\mu} = \frac{1}{2^n} \binom{n}{\mu}, \quad a'_{\mu\nu} \begin{cases} = \nu & \text{for } \mu = \nu, \\ = -\nu & \text{for } \mu = \nu + 1, \\ = 0 & \text{otherwise,} \end{cases}$$

and the left side of (4) becomes

⁷ See for instance Wilansky [8].

⁸ See for instance Hardy [2, p. 213].

$$T_n = \sum_{\nu=1}^n \nu^{-1/2} \left| \nu \frac{1}{2^n} \binom{n}{\nu} - \nu \frac{1}{2^n} \binom{n}{\nu+1} \right|.$$

This can be rewritten in the form

$$T_n = O(1) \sum_{\nu=1}^n \nu^{1/2} \left| \frac{1}{2^n} \binom{n}{\nu} - \frac{1}{2^n} \binom{n}{\nu+1} \right|,$$

from which we obtain

$$T_n = n^{1/2} \cdot \max_{0 \leq \nu \leq n} \frac{1}{2^n} \binom{n}{\nu} O(1) = O(1).$$

Corollary 3 and Theorem 4 combine to

COROLLARY 4. *Let the sequence $\{s_n\}$ be (C, k) -summable ($k > 1$) and let*

$$\sigma_n^k = s + o(n^{(1-2k)/2}),$$

then the sequence is Euler-summable.

4. The investigations of the previous section can be extended to the case of absolute summability. One says that the sequence $\{s_n\}$ is absolutely A -summable, if

$$\sum_{n=0}^{\infty} |\sigma_n - \sigma_{n+1}| < \infty.$$

Also in use is the notation: $\sigma_n = a(\Omega(n))$ if

$$\sum_{n=0}^{\infty} |\sigma_n \Omega^{-1}(n) - \sigma_{n+1} \Omega^{-1}(n+1)| < \infty.$$

With this convention and using the notation of the proof of Theorem 2 we can state

THEOREM 5. *The sequence $\{s_n\}$ is absolutely B -summable if*

$$\sigma_n = a(\Omega(n))$$

and if

$$\sum_{n=0}^{\infty} \sum_{\nu=0}^n |c_{n\nu} - c_{n+1,\nu}| < \infty.$$

The proof is along the lines of the proof of Theorem 2. Application of this theorem shows, that Corollaries 3 and 4 still hold, if reworded for absolute summability.

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ON EXTENSIONS OF THE FIELD OF CONSTANTS OF AN ALGEBRAIC FUNCTION FIELD

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1. The purpose of this paper is to discuss the applications of a previous paper, *Reduction of an algebraic function field modulo a prime in the constant field*, [3], to cases where the constant field is extended. For completeness, we shall restate the principal results of that paper.

Let K be an algebraic function field of one variable with constant field k , and let k have a discrete non-Archimedean valuation h . Let x be a fixed arbitrary element in K transcendental over k . Let h be extended to a valuation, also denoted by h , of $k(x) = F$ by the Gaussian definition. Then extend h to valuations H_i , $i = 1, 2, \dots, s$, on K . Let K_i be the residue class field of K modulo H_i . Let $\bar{k}$ and $\bar{F}$ denote the residue class fields of k and F modulo h . Properly interpreted, K_i is finite algebraic over $\bar{k}(x) = \bar{F}$, i.e., K_i is an algebraic function field. If k_i is the constant field of K_i , then k_i is of finite degree r_i over $\bar{k}$. Let g be the genus of K and g_i the genus of K_i . Let $e(H_i)$ be the ramification index of H_i . If the radical degree of each H_i is 1, then

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