ON ENTIRE FUNCTIONS DEFINED BY A DIRICHLET SERIES¹

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1. **Introduction.** Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-s\lambda_n}, \quad \lambda_{n+1} > \lambda_n, \quad \lambda_1 \geq 0, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty, \quad s = \sigma + it,$$

where

$$(1) \quad \liminf_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = h > 0.$$

Let σ_c and σ_a denote, respectively, the abscissa of convergence and the abscissa of absolute convergence of $f(s)$. If $\sigma_c = -\infty$, $f(s)$ defines an *entire function*. I shall suppose throughout that $\sigma_c = \sigma_a = -\infty$.

Let

$$\mu(\sigma) = \max_{n \geq 1} |a_n| e^{-\sigma\lambda_n}, \quad M(\sigma) = \max_{-\infty < t < \infty} |f(\sigma + it)|.$$

It follows [1, p. 187] from hypothesis (1) that² $D < \infty$ and

$$(2) \quad \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = 0.$$

For a positive ϵ , one can choose therefore a positive integer $N(\epsilon)$ such that $\log n < (\epsilon/2)\lambda_n$ for $n \geq N(\epsilon)$. Consequently

$$\begin{aligned} M(\sigma) &\leq \sum_{n=1}^{\infty} |a_n| e^{-\sigma\lambda_n} \\ &= \sum_{n=1}^{N(\epsilon)-1} |a_n| e^{-\sigma\lambda_n} + \sum_{n=N(\epsilon)}^{\infty} |a_n| e^{-(\sigma-\epsilon)\lambda_n} \cdot e^{-\epsilon\lambda_n}, \\ &< N(\epsilon)\mu(\sigma) + \mu(\sigma - \epsilon) \sum_{n=N(\epsilon)}^{\infty} \frac{1}{n^2}, \\ (3) \quad &< K_1\mu(\sigma - \epsilon), \quad \sigma > \sigma_a. \end{aligned}$$

For a fixed t_0 , let $S(R)$ denote the horizontal strip $|t - t_0| < R$. Put

$$M_S(\sigma) = \max_{|t - t_0| \leq R} |f(\sigma + it)|$$

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² For notations see [1].

and let

$$\lim_{\sigma \rightarrow -\infty} \sup \frac{\log \log M(\sigma)}{\inf -\sigma} = \frac{\rho}{\lambda}; \quad \lim_{\sigma \rightarrow -\infty} \sup \frac{\log \log M_S(\sigma)}{\inf -\sigma} = \frac{\rho_S}{\lambda_S}.$$

For $0 < \rho < \infty$ and $0 < \rho_S < \infty$, respectively, let

$$\lim_{\sigma \rightarrow -\infty} \sup \frac{\log M(\sigma)}{\inf e^{-\rho\sigma}} = \frac{T}{\tau}; \quad \lim_{\sigma \rightarrow -\infty} \sup \frac{\log M_S(\sigma)}{\inf e^{-\rho_S\sigma}} = \frac{T_S}{\tau_S}.$$

Mandelbrojt and Gergen [1, pp. 219–220] have proved that the order ρ_S of $f(s)$ in each horizontal strip $S(\pi a)$, with $a > D$, is equal to the order ρ of $f(s)$. I prove here the following:

THEOREM 1. *The lower order λ_S of $f(s)$ in each horizontal strip $S(\pi a)$, with $a > D$, is equal to the lower order λ of $f(s)$.*

THEOREM 2. *If $h = \infty$, then the type T_S and the lower type τ_S of $f(s)$ in each horizontal strip $S(\pi a)$, with $a > 0$, satisfy respectively*

$$T_S \geq e^{-\pi \rho a} T; \quad \tau_S \geq e^{-\pi \rho a} \tau.$$

2. PROOF OF THEOREM 1. If $S(\pi a)$ is the strip $|t - t_0| < \pi a$, and if

$$M_1(\sigma_j + it_0) = \text{Max } |f(s)|$$

when $s \in C(\sigma_j + it_0, \pi a)$, we see [1, Theorem XVI] that, for an arbitrary σ_j :

$$(4) \quad \log M_1(\sigma_j + it_0) \geq \log |a_n| - \log \Lambda_n - \lambda_n \sigma_j - K, \quad (n \geq 1),$$

where K is a constant. Since $D < \infty$, we have [1, Theorem XVII]

$$\limsup_{n \rightarrow \infty} \frac{\log \Lambda_n}{\lambda_n} < \infty,$$

that is to say:

$$(5) \quad \log \Lambda_n < P \lambda_n, \quad (n \geq 1),$$

where P is a constant.

Let us denote by $s_j^* = \sigma_j^* + it_0^*$ a quantity such that $M_1(\sigma_j + it_0) = |f(s_j^*)|$. It is obvious that $M_S(\sigma_j^*) \geq M_1(\sigma_j + it_0)$ and that $\sigma_j^* = \sigma_j + k_j$, where $|k_j| \leq \pi a$. The inequality (4) thus allows us to write

$$\begin{aligned} \log M_S(\sigma_j^*) &= \log M_S(\sigma_j + k_j) \\ &\geq \log M_1(\sigma_j + it_0) \\ &\geq \log |a_n| - P \lambda_n - \lambda_n \sigma_j - K, \quad (n \geq 1), \\ &= \log (|a_n| e^{-(\sigma_j + P) \lambda_n}) - K, \quad (n \geq 1). \end{aligned}$$

Consequently

$\log MS(\sigma_j^*) \geq \log \mu(\sigma_j + P) - K > \log M(\sigma_j + P + \epsilon) - \log K_1 - K$
by (3), and we get $\lambda_S \geq \lambda$. But obviously $\lambda_S \leq \lambda$, and therefore $\lambda_S = \lambda$.

PROOF OF THEOREM 2. By the theorem of Mandelbrojt and Gergen [loc. cit.], since the conditions of that theorem are satisfied:

$$\rho_S = \rho.$$

If $h = \infty$, then [1, Theorem XVII]

$$\limsup_{n \rightarrow \infty} \frac{\log \Lambda_n}{\lambda_n} = 0,$$

and for a positive ϵ'

$$\log \Lambda_n < \epsilon' \lambda_n, \quad (n \geq 1).$$

Hence from (4), we have

$$\begin{aligned} \log M_S(\sigma_j^*) &= \log M_S(\sigma_j + K_j) \geq \log M_1(\sigma_j + it_0) \\ &\geq \log |a_n| - \epsilon' \lambda_n - \lambda_n \sigma_j - K, \quad (n \geq 1) \\ &= \log (|a_n| e^{-(\sigma_j + \epsilon') \lambda_n}) - K, \quad (n \geq 1). \end{aligned}$$

Consequently

$$\begin{aligned} \log M_S(\sigma_j^*) &= \log M_S(\sigma_j + k_j) \geq \log \mu(\sigma_j + \epsilon') - K \\ &> \log M(\sigma_j + \epsilon' + \epsilon) - \log K_1 - K \end{aligned}$$

by (3), and we get

$$\begin{aligned} \frac{\log M_S(\sigma_j^*)}{e^{-\rho S \sigma_j^*}} &= \frac{\log M_S(\sigma_j + k_j)}{e^{-\rho S(\sigma_j + k_j)}} > \frac{\log M(\sigma_j + \epsilon' + \epsilon)}{e^{-\rho(\sigma_j + k_j)}} - \frac{\log K_1 + K}{e^{-\rho(\sigma_j + k_j)}} \\ &= e^{\rho(\sigma_j - \epsilon' - \epsilon)} \frac{\log M(\sigma_j + \epsilon' + \epsilon)}{e^{-\rho(\sigma_j + \epsilon' + \epsilon)}} - \frac{\log K_1 + K}{e^{-\rho(\sigma_j + k_j)}}. \end{aligned}$$

Hence

$$T_S \geq e^{\rho(k_j - \epsilon' - \epsilon)} T; \quad \tau_S \geq e^{\rho(k_j - \epsilon' - \epsilon)} \tau,$$

and, since $|k_j| \leq \pi a$ and ϵ', ϵ are arbitrary, we have

$$T_S \geq e^{-\pi \rho a} T; \quad \tau_S \geq e^{-\pi \rho a} \tau.$$

REFERENCE

1. S. Mandelbrojt, *Dirichlet series*, Rice Inst. Pamphlet, vol. 31, no. 4 (1944).

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