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QUASI-NIL RINGS

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Rings have been studied which have among others the property that every commutator $xy - yx$ is in the nucleus. It seems appropriate to consider rings in which the square of every element is in the nucleus, a property that is shared by both associative and Lie rings. Under the additional assumptions of primeness and characteristic different from 2 it can be shown that such rings are either associative or have the property that $x^2 = 0$, for every element x of the ring. If further $(x, y, z) + (y, z, x) + (z, x, y)$ is in the nucleus for all elements x, y, z of the ring, then the ring is either associative or a Lie ring.

We use the notation $(x, y, z) = (xy)z - x(yz)$. The nucleus N of a ring R consists of all $n \in R$ such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$. N is a subring of R .

LEMMA. *Let R be a prime ring satisfying $x^2 \in N$ for every $x \in R$ and of characteristic different from 2. Then either R is associative or $N^2 = 0$.*

PROOF. For all $r, s \in R$, $rs + sr = (r + s)^2 - r^2 - s^2$ must be in N . Select $n, n' \in N$, and $x, y, z \in R$. Then $(n(n'x + xn'), y, z) = 0$, so that $(nn'x, y, z) = -(nxn', y, z)$. Similarly $(nxn', y, z) = -(xnn', y, z)$ and $(xnn', y, z) = -(nn'x, y, z)$. By combining these three equalities it follows that $2(nn'x, y, z) = 0$. Assuming characteristic not 2 it then follows that $(nn'x, y, z) = 0$. Since $(nx, y, z) = ((nx)y)z - (nx)(yz) = (n(xy))z - n(x(yz)) = n((xy)z) - n(x(yz)) = n(x, y, z)$, we replace n

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by nn' and obtain $(nn'x, y, z) = nn'(x, y, z)$. We deduce that $nn'(x, y, z) = 0$, so that $N^2(R, R, R) = 0$. Let I be the ideal generated by N^2 , and J the ideal generated by all associators (R, R, R) . We note that $nn'x = n(n'x + xn') - (nx + xn)n' + xnn'$, so that $N^2R \subset RN^2 + N^2$. Consequently $I = RN^2 + N^2$. In an arbitrary ring $J = (R, R, R) + (R, R, R)R$. It follows from $N^2(R, R, R) = 0$, that $IJ = 0$. Since R is prime either $I = 0$, or $J = 0$. If $I = 0$, then $N^2 = 0$. On the other hand if $J = 0$, then R is associative. This completes the proof of the lemma.

THEOREM 1. *Let R be a prime ring of characteristic not 2 satisfying $r^2 \in N$ for all $r \in R$. Then either R is associative or $r^2 = 0$, for all $r \in R$.*

PROOF. Let us consider the case $N \neq R$. Then it follows from the lemma that $N^2 = 0$. Let $K = N + NR$. Since $rn = (rn + nr) - nr \in K$ and $snr = (sn + ns)r - nsr \in K$, for all $n \in N$ and $r, s \in R$, K must be an ideal of R . Moreover if $n' \in N$, $nrn' = n(rn' + n'r) - nn'r = 0$, since $N^2 = 0$. This suffices to show $K^2 = 0$. But R is prime and so $K = 0$. But then $N = 0$, whence $r^2 = 0$, for all $r \in R$. This completes the proof of the theorem.

THEOREM 2. *Let R be a prime ring of characteristic not 2 satisfying (i) $x^2 \in N$ for all $x \in R$, and (ii) $(x, y, z) + (y, z, x) + (z, x, y) \in N$ for all $x, y, z \in R$. Then R is either associative or a Lie ring. Conversely all associative rings and all Lie rings satisfy both (i) and (ii).*

PROOF. Assume that R satisfies (i) and (ii), and suppose $N \neq R$. Then Theorem 1 implies that $x^2 = 0$, for all $x \in R$, and consequently R is anti-commutative. For any $n \in N$, $nxy = -xny = xyn$, and also $nxy = -xyn$, so that $2nxy = 0$. Thus $NR^2 = 0$. The set T of all $t \in R$ such that $tR = 0$, forms an ideal of R which must be zero since R is prime. But since $NR \subset T$ we obtain first $NR = 0$, and subsequently $N = 0$. In any anti-commutative ring $(x, y, z) + (y, z, x) + (z, x, y) = (xy)z - x(yz) + (yz)x - y(zx) + (zx)y - z(xy) = 2((xy)z + (yz)x + (zx)y)$ which equals twice the Jacobian of x, y, z . Then because of (ii) we conclude that the well known Jacobi identity holds and thus R is a Lie ring. The converse follows automatically. In an associative ring $N = R$, so that (i) and (ii) are trivially satisfied. All Lie rings are anti-commutative and satisfy the Jacobi identity, so that (ii) follows. This completes the proof of the theorem.

N. H. McCoy [1] and R. L. SanSoucie [2] have shown independently that any primitive ring is prime. Consequently the above theorems may be extended to primitive and semi-simple rings in the usual way.

We conclude with a couple of examples.

EXAMPLE 1. Let x, y, z, n be basis elements of the algebra A over an arbitrary field F . All products of basis elements are defined to be zero with the exception of $xy = -yx = z$, $zx = -xz = y$, $yz = -zy = x$, and $n^2 = n$. For every $\alpha, \beta, \gamma, \delta \in F$, $(\alpha x + \beta y + \gamma z + \delta n)^2 = \delta^2 n$ is in the nucleus. Thus for every $r \in R$, $r^2 \in N$. R has four ideals, the trivial ones, the ideal B generated by x, y, z , and the ideal C generated by n . Also $BC = 0 = CB$, while $B^2 = B$, and $C^2 = C$.

EXAMPLE 2. Let $1, x, y$ be basis elements of the algebra R over an arbitrary field F , where $xy = 1$, $yx = x^2 = y^2 = 0$. For any $\alpha, \beta, \gamma \in F$, $(\alpha + \beta x + \gamma y)^2 = \alpha^2 + 2\alpha\beta x + 2\alpha\gamma y + \beta\gamma = 2\alpha(\alpha + \beta x + \gamma y) + \beta\gamma - \alpha^2$. Thus R is quadratic over F . Moreover it can be readily verified that R is simple, power-associative, and that all commutators of R are contained in F . R is not associative since $(x, y, y) = y$. Also $(x + y)^2 = 1 \neq 0$. If F happens to be a field of characteristic 2 then $r^2 \in F$ for every $r \in R$. We see that Theorem 1 fails to hold for rings of characteristic 2.

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