A NOTE ON PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITHOUT DAMPING

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We consider the differential equation

$$(1) \quad \ddot{x} + g(x) = p(t),$$

where, in addition to satisfying conditions which insure existence and uniqueness of solutions, $g(x)$ and $p(t)$ satisfy the following conditions:

- (i) there exist positive constants τ and k such that for all t , $p(t) = p(t+\tau)$ and $|p(t)| < k$;
- (ii) $g(x)$ is continuous, increasing, $g(0) = 0$, and for $|x|$ sufficiently large, $|g(x)| > k$.

We show that for τ sufficiently small, Equation (1) has a solution $x(t)$ of period τ . Our method of proof consists of showing that a certain mapping of the phase plane of Equation (1) has a fixed point, not by use of the Brouwer theorem, but by a somewhat more general result concerning the index of the bounding curve of a simply-connected region relative to a vector field induced by the mapping; cf. [1, p. 337].

We consider Equation (1) in terms of the system:

$$(2) \quad \dot{x} = y, \quad \dot{y} = p(t) - g(x),$$

the solutions $(x(t), y(t))$ of which define the so-called phase trajectories of the system. Let $x=a$ be the root of $g(x) = -k$, and $x=b$ be the root of $g(x) = k$. That these roots exist and are unique follows immediately from (ii). We define

$$G(x) = \int_0^x g(v) dv$$

and using (ii), we observe that the graph of $y = G(x)$ has the following properties:

- (iii) it is tangent to the x -axis at the origin and is everywhere concave up;
- (iv) its slope $G'(x) = g(x)$ is such that $-k < G'(x) < k$ for $a < x < b$, $G'(x) > k$ for $x > b$, and $G'(x) < -k$ for $x < a$.

Hence, there clearly exists a positive constant C_1 such that the line $y = kx + C_1$ intersects the graph of $y = G(x)$ at exactly two points whose abscissas r_2 and r_3 are such that $r_2 < a < b < r_3$. Similarly there exists a positive constant C_2 such that the line $y = -kx + C_2$ intersects the graph of $y = G(x)$ at exactly two points whose abscissas r_1 and r_4 are such that $r_1 < r_2 < r_3 < r_4$.

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We denote by Γ_1 the open arc defined by $y^2/2 + G(x) = kx + C_1$, $y > 0$, by Γ_2 the open arc defined by $y^2/2 + G(x) = -kx + C_2$, $y < 0$, and observe that the end points of these arcs are at $(r_2, 0)$, $(r_3, 0)$ and $(r_1, 0)$, $(r_4, 0)$ respectively.

LEMMA 1. *The slope of any phase trajectory of (2) at a point of either Γ_1 or Γ_2 is less than the slope of the curve Γ_1 or Γ_2 at that point.*

PROOF. We prove the lemma only for a point (x_0, y_0) on Γ_1 ; the proof for a point on Γ_2 is entirely similar, and is omitted. Clearly the slope of Γ_1 at (x_0, y_0) is given by $(k - g(x_0))/y_0$, while the slope of the trajectory of (2) at (x_0, y_0) for $t = t_0$ is given by $(p(t_0) - g(x_0))/y_0$, which is clearly less than the slope of Γ_1 there. This proves the lemma.

We now denote by P_1 , P_2 , and P_3 the points $(r_1, 0)$ and $(r_3, 0)$ respectively; by Q the intersection of the line $x = r_3$ with Γ_2 , and by R the closed region bounded by the curves Γ_1 , Γ_2 from P_1 to Q , and the line segments P_1P_2 , P_3Q . We now prove another simple

LEMMA 2. *Let (x_0, y_0) be a point on the boundary of R , and let $(x(t), y(t))$ be the solution of (2) for which $x(0) = x_0$, $y(0) = y_0$. Then the trajectory of this solution can only leave R for some $t_0 \geq 0$, if $r_1 \leq x(t_0) \leq r_2$, and $y(t_0) = 0$.*

PROOF. Due to Lemma 1, we need only prove that if (x_0, y_0) is on the segment P_3Q , then the trajectory at (x_0, y_0) for some arbitrary $t = t_0 > 0$ passes first into R as t increases from t_0 . This is obvious if (x_0, y_0) is Q since Q is on Γ_2 ; it is also obvious for (x_0, y_0) between P_3 and Q since then $y_0 < 0$, and hence $\dot{x}(t_0) = y(t_0) < 0$. Finally if (x_0, y_0) is P_3 , then since $y(t_0) = -g(x(t_0)) + p(t_0) < 0$, $y(t) < 0$ for $t > t_0$, $t - t_0$ sufficiently small; hence $x(t) < x(t_0)$ also for $t > t_0$, $t - t_0$ sufficiently small.

This essentially proves the lemma.

We now define $M = |2(G(a) + ka - C_2)|^{1/2}$, $m = (2(G(a) - ka - C_1))^{1/2}$, $d = a - r_2$, and $k = \max_{r_1 \leq x \leq 0} (-g(x) + k)$ and prove the following:

THEOREM. *In terms of the definitions above, if $\tau < \min(d/M, m/K)$, then system (2), hence Equation (1), has a solution of period τ .*

PROOF. We define the mapping T of the (x, y) plane into itself as follows: $T(x_0, y_0) = (x_1, y_1)$ where $x_1 = x(\tau)$, $y_1 = y(\tau)$, and $(x(t), y(t))$ is the solution of (2) such that $x(0) = 0$, $y(0) = 0$. We show that this mapping has a fixed point; this will prove the theorem; cf. [1, p. 270].

We consider the vector field defined by this mapping, and will show that the index of the boundary of R relative of this field (cf. [1, p. 337]) is $+1$; this will prove that R contains a critical point of this field; i.e., that the mapping T has a fixed point in R .

Let (x_0, y_0) be a point of Γ_1 ; we show first that $T(x_0, y_0)$ is in R . For if not, there exists by Lemma 2 a smallest t_1 , $0 < t_1 < \tau$, such that $y(t_1) = 0$, $r_1 \leq x(t_1) \leq r_2$; here as throughout this proof $(x(t), y(t))$ is the solution of (2) used in the definition of T . Suppose first $x_0 \geq a$; then clearly for $0 < t < t_1$, $|y(t)| < M$. Since $\dot{x}(t) = y(t)$,

$$|x(t_1) - x_0| = \left| \int_0^{t_1} y(t) dt \right| < M\tau.$$

However, $|x(t_1) - x_0| \geq a - r_2 = d$; i.e., $d < \tau M$, a contradiction. If $r_2 < x_0 < a$, then since $\dot{y} = -g(x) + p(t) > 0$ for $x = x(t) < a$, there exists a $t_0 > 0$ such that $(x(t), y(t))$ is in R for $0 \leq t \leq t_0$, and $x(t_0) = a$. From here, the argument toward a contradiction proceeds as above, and we omit it. In fact, if (x_0, y_0) is on either the segment P_3Q or the curve Γ_2 from $(a, -M)$ to Q , the same argument, in essence, applies to show that $T(x_0, y_0)$ is in R .

If (x_0, y_0) is on Γ_2 while $x < a$, or on the segment P_1P_2 , then if $T(x_0, y_0) = (x_1, y_1)$ is not in R , we show finally that $r_1 \leq x_1 \leq a$, $0 \leq y_1 \leq m$. In any case, there exists t_0 , $0 \leq t_0 < \tau$, such that $r_1 \leq x(t_0) \leq r_2$, $y(t_0) = 0$. If $x(t) < a$ for $t_0 < t < \tau$, then clearly $x(t) > r_1$ for these values of t . Assume that for t_1 , $t_0 < t_1 < \tau$, we have $y(t_1) > m$. Then since $\dot{y} = -g(x) + p(t)$, we have

$$y(t_1) - y(t_0) = \int_{t_0}^{t_1} (-g(x(t)) + p(t)) dt < K\tau.$$

But $y(t_1) - y(t_0) = y(t_1) > m$; hence $m < K\tau$, a contradiction, and we conclude that $x(t_2) = a$ for some t_2 , $t_0 < t_2 < \tau$, while $0 < y(t_2) < m$. Since $M > m$, and $x(t_2) - x(t_0) > d$, we easily arrive at a contradiction as before by integrating $\dot{x}(t) = y(t)$ from t_0 to t_2 ; we omit the details.

If the vector field $(x, y) \rightarrow T(x, y)$ has a critical point on the boundary of R , then by definition, the mapping has a fixed point there and there is nothing more to prove. We therefore assume that as (x, y) moves in a counter-clockwise circuit around the boundary of R , the vector at each point of the boundary is nonzero; since it is also a continuous function of (x, y) and in view of the location of the terminal point $T(x, y)$ for (x, y) on the boundary of R as established in the proof of this theorem, it follows that the index of this circuit relative to this field is $+1$. This completes the proof of the theorem.

REFERENCE

1. S. Lefschetz, *Differential equations: Geometric theory*, Interscience Publishers, Inc., New York, 1957.