SOME SEMIGROUPS ON THE TWO-CELL

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A topological semigroup is a nonvoid, Hausdorff space, S, together with a continuous, associative multiplication defined on S. Here the term semigroup will always denote a topological semigroup. In [**4**], Wallace and Koch have shown that if the circle, B, is a semigroup such that $B^2=B$, then either B is a group, or the multiplication on B is of the trivial kind $xy=x$, or $xy=y$. In [**6**], Mostert and Shields give a description of a semigroup on the two-cell where the boundary of the two-cell relative to the plane is a group. The main results of this paper will be a description of a semigroup on the two-cell where the multiplication satisfies the following two conditions: one; for x and y in the boundary of the two-cell relative to the plane $xy=x$, and two; there is a zero but no other idempotent in the interior of the two-cell. The following theorem will be proved:

THEOREM A. *Let S, topologically the two-cell, be a semigroup with zero in the interior. Assume for x and y in B, the boundary of S relative to the plane, $xy=x$. Suppose, moreover, that S has no idempotents, except zero, in the interior. Then there exists an I-semigroup T in S such that $S=BT$. Also for e and f in B, t and s in T $(es)(ft)=e(st)$; and if $es=ft$ then $s=t$.*

There is, as will be obvious from the proof, a left-right dual of this theorem. Due to the length of the proof of this theorem, which will be given in a sequence of lemmas and theorems, some definitions and background material have been omitted. The reader is referred to [**6**] for any definitions and results not given here.

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A semigroup P will be said to satisfy condition (α) if (i) there exists a set A in P such that $A\subseteq f(A)$, where f is the function defined by $f(x)=x^2$ for all x in P; and (ii) there exists y in A such that $y\neq y^2$. In the following discussion P will be assumed to be a compact semigroup which satisfies condition (α). For an element p in A such that

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$p \neq p^2$, $s(p)$ will denote a fixed sequence of elements p_n, $n=0, 1, 2, \cdots$ with the properties that (i) $p_0=p$, and (ii) $p_n^2=p_{n-1}$. For positive integers n and j, let $Z_{n,j}(s(p))$ be defined as follows:

$$Z_{n,j}(s(p)) = \left\{ p_m^i : \frac{i}{2^m} \leqq \frac{j}{2^n} \text{ and } p_m \in s(p) \right\}^*,$$

where A^* denotes the topological closure of a set A. For real, nonnegative t, let $Z_t(s(p)) = \bigcap \{ Z_{n,j}(s(p)) : j/2^n > t \}$. Also let $\delta(Z_t(s(p))) = Z_t(s(p)) \setminus \bigcup_{r<t} Z_r(s(p))$, where $A \setminus B$ denotes the set theoretic difference of A by B. In the following discussion consider the element p in A and the sequence $s(p)$ fixed and let $Z_0(s(p))$ be denoted by Z.

LEMMA 1. *$Z = \bigcap_{t>0} \{Z_t(s(p))\}$ and Z is a compact, commutative group.*

PROOF. Clearly $Z = \bigcap_{t>0} \{Z_t(s(p))\}$ and Z is nonvoid since P is compact and Z is the intersection of a tower of closed subsets of P. Since Z is closed it is also compact. Using the fact that multiplication is continuous in P and that the sequence $s(p)$ was chosen so that $p_n^2 = p_{n-1}$, it can be easily shown that Z is a commutative subsemigroup of P and for a in Z, p_n in $s(p)$ that $p_n \in aP \cap Pa$. From this last fact it follows that any idempotent in Z acts as an identity for the sequence $s(p)$, and therefore for all of Z. Since Z is a compact subsemigroup of P it contains at least one idempotent. But any idempotent in Z acts as an identity, hence Z can contain at most one idempotent. Thus Z is a group, [**9**].

LEMMA 2. *If y and z belong to $\delta(Z_t(s(p)))$, there exists g in Z such that $y=gz$.*

PROOF. Since y and z are in $\delta(Z_t(s(p)))$, $y = \lim p_{m_n}^{i_n}$ and $z = \lim p_{k_n}^{j_n}$ where p_{m_n} and p_{k_n} are subsequences of $s(p)$ and $i_n/2^{m_n} \to t^+ \leftarrow j_n/2^{k_n}$. We may assume that $i_n/2^{m_n} \geqq j_n/2^{k_n}$ for all n. Let $r_n/2^{q_n} = i_n/2^{m_n} - j_n/2^{k_n}$. Now $r_n/2^{q_n}$ converges to zero, therefore $\{p_{q_n}^{r_n}\}$ clusters in Z. Since a subsequence of $\{p_{q_n}^{r_n}\}$ converges to some g in Z, let us assume that the sequence converges to g. Hence we have, $y = \lim p_{m_n}^{i_n} = (\lim p_{q_n}^{r_n})(\lim p_{k_n}^{j_n}) = gz$.

THEOREM 1. *Let P be a compact semigroup satisfying condition (α). For p in A such that $p \neq p^2$ let $Z(s(p))$ be defined as above. Then there exists a one-parameter semigroup σ from $[0, 1]$ to $Z_1(s(p))$, such that $\sigma(0)=e$, the identity of $Z(s(p))$, and $\sigma(1)g=p$ for some g in $Z(s(p))$.*

PROOF. The proof of this theorem is long and is divided into a sequence of short steps.

(i) Since $e\in Z(s(p))$, there is a sequence $\{p_{m_n}^{i_n}\}$ such that $e=\lim p_{m_n}^{i_n}$ and $i_n/2^{m_n}$ converges to zero. For notational reasons, let $r_n=i_n/2^{m_n}$ and $p_{m_n}^{i_n}=p^{r_n}$. For each n, define $\theta(n)=[1/r_n]$, where $[a]$ is the least integer greater than or equal to a. From the definition of $\theta(n)$ and the fact that $r_n\to 0$, it follows that $\theta(n)r_n\to 1^+$.

For each real number s, such that $0<s\leqq 1$, define a function f_s from J, the positive integers, to $Z_1(s(p))$ by $f_s(n)=p^{[s\theta(n)]r_n}$. For each s, f_s is a continuous function and therefore has a unique extension, f_s^*, to $\beta(J)$, the Stone-Čech compactification of J. Let $\alpha\in\beta(J)\backslash J$ be chosen and considered fixed in the following discussion.

For each real s, $0<s\leqq 1$, define $\tau(s)=f_s^*(\alpha)$ and $\tau(0)=e$. From the definition of τ it can be shown that for s, t and $s+t$ in $[0, 1]$, $\tau(s+t)=\tau(s)\tau(t)$. Also from the definition of τ and the fact that $[\theta(n)t]r_n\to t^+$, it follows that $\tau(t)\in Z_t(s(p))$ and $\tau(t)$ does not belong to $Z_r(s(p))$ for any $r<t$, that is, $\tau(t)\in\delta(Z_t(s(p)))$ for $0<t\leqq 1$.

(ii) G, defined by $G=\bigcap_{t>0}\{(\tau[0, t])^*\}$, is a group with identity $e=\tau(0)$. Clearly $e\in G$ and $G\subseteq Z(s(p))$. Thus it remains to show that $G^2\subseteq G$, since G is compact and contains only one idempotent which acts as an identity. By a straightforward argument using the facts that multiplication is continuous in P and that for s, t and $s+t$ in $[0, 1]$, $\tau(s+t)=\tau(s)\tau(t)$, it follows immediately that $G^2\subseteq G$.

(iii) Let Q be defined by $Q=(\tau[0, 1/2))^*$. Q is a compact, adequate local semigroup with maximal subgroup G. This statement follows almost immediately from the facts that if x is in Q then $x=\tau(a)g$ for some a such that $0\leqq a\leqq 1/2$, and some $g\in G$. To show the latter let us assume that x belongs to Q. Then there is a sequence a_n in $[0, 1/2]$ such that $x=\lim \tau(a_n)$. Since $a_n\in[0, 1/2)$ for each n, there is a real number t in $[0, 1/2]$ such that $t=\lim a_m$, where $a_m=a_{n_j}$ is some subsequence of a_n. We may assume that $a_m\leqq t$ for each m, or $a_m\geqq t$ for each m. If $a_m\leqq t$, let $b_m=t-a_m$. Then $\tau(t)=\tau(a_m)\tau(b_m)$ and $b_m\to 0$. Hence $\tau(t)=\lim\tau(a_m)\lim\tau(b_m)$. That is $\tau(t)=xg$ for some $g\in G$. Thus $x=\tau(t)g^{-1}$. The case for $a_m\geqq t$, for each m, is similar.

Now suppose x and y are in Q and $xy=e$. By the preceding argument $x=\tau(a)g$ and $y=\tau(b)h$ for some a and b in $[0, 1/2]$, g and h in G. Since $xy=e$, by (ii) we may conclude that $a+b=0$, hence $a=b=0$. Thus $x=g$ and $y=h$ and both belong to G. This concludes the proof of the above statement.

It is not difficult to show that Q contains no idempotents except $\tau(0)$, hence by [5], there is a one-parameter semigroup γ from $[0, 1]$ to Q such that $\gamma(t)\in G$ if and only if $t=0$. Reparametrize γ so that $\gamma(t)/G=\tau(t)/G$ and denote the new one-parameter semigroup by σ. This gives the first part of Theorem 1. To obtain $p=\sigma(1)g$ for some

g in $Z(s(p))$, we need only to note that p and $\sigma(1)$ both belong to $\delta(Z_1(s(p)))$ and apply Lemma 2.

PROOF OF THEOREM A. Throughout this proof S will denote a semigroup satisfying the hypothesis of Theorem A.

LEMMA 3. *For each a in S, there exists an x in S such that $x^2=a$.*

PROOF. Define $f: S\rightarrow S$ by $f(x)=x^2$. f is a continuous function and for each e in B, the boundary of S, $f(e)=e^2=e$. If f does not map S onto S, then it would be possible to retract S onto B which presents a contradiction. Thus f maps S onto S and the lemma is proved.

Let $p\neq 0$ be in S^0, the interior of S, and let $s(p)$ be a fixed sequence of 2^nth roots of p as defined above. Let e denote the identity of $Z(s(p))$, which, by the hypothesis on S, belongs to B. Denote by σ the one-parameter semigroup as constructed in Theorem 1 such that $\sigma(0)=e$.

LEMMA 4. *For t in $(1, \infty)$, define $\tau(t)=\sigma(1)^n\sigma(a)$, where $t=n+a$, n an integer and $0\leqq a<1$. For t in $[0, 1]$ define $\tau(t)=\sigma(t)$. τ so defined is a continuous homomorphism from R, the additive semigroup of reals $\geqq 0$, into S and $(\tau[0, \infty))^*$ is a compact, connected, commutative subsemigroup with identity e.*

PROOF. The proof of this lemma makes frequent use of the fact that $\sigma(1)=\sigma(t)\sigma(1-t)$ for any t in $[0, 1]$. Using this it is easy to see that $\sigma(1)^n\sigma(t)=\sigma(t)\sigma(1)^n$ and hence from the definition of τ, that $\tau(a+b)=\tau(a)\tau(b)$ for all real a and b. This proves that τ is a homomorphism from R into S. Since $\tau[0, \infty)$ is a commutative, connected subsemigroup of S, it follows that $(\tau[0, \infty))^*$ is again one such.

In the following lemma, $J(x) = x\cup xS\cup Sx\cup SxS$, $J_x=\{y: J(y)=J(x)\}$ and $I(x)=J(x)\backslash J_x$.

LEMMA 5. *Let C be a continuum that is a semigroup with zero. Let $A=A^*$ be a subset of C such that for some x_0 in C the following holds: (1) $A^*\backslash A^0\subseteq J_{x_0}$; (2) $J_{x_0}\neq\{x_0\}$; and (3) A^0 is nonvoid and is contained in $J(x_0)$. Then the zero for C belongs to A^0.*

PROOF. In [9], Wallace proves that if C is a continuum and $J_{x_0}\neq\{x_0\}$ then $I(x_0)^*=J(x_0)$ and $I(x_0)$ is connected. By the above hypothesis on A we have $I(x_0)\cap A=(I(x_0)\cap A^0)\cup(I(x_0)\cap A^*\backslash A^0)=I(x_0)\cap A^0$. This implies that $I(x_0)\cap A^0$ is both open and closed in $I(x_0)$, a connected set. Thus either $I(x_0)\cap A^0$ is void or $I(x_0)\cap A^0=I(x_0)$. If $I(x_0)\cap A^0$ is void, since $J(x_0)=I(x_0)^*$, we have $J(x_0)\subseteq C\backslash A^0$, a contradiction by condition 3. Hence the zero for C belongs to $I(x_0)$ which is contained in A^0 and the lemma is proved.

COROLLARY. *Let C be a continuum that is a semigroup with zero. Let $A = A^*$ be a subset of C such that*: (1) $A^* \backslash A^0$ *is a nontrivial group with identity* u; *and* (2) A^0 *is nonvoid and contained in* $J(u)$. *Then the zero for C belongs to* A^0.

PROOF. This follows immediately from the above lemma letting $x_0 = u$.

LEMMA 6. *Let S be as defined above. The zero of S belongs to* $(\tau[0, \infty))^*$.

PROOF. Let $(\tau[0, \infty))^*$ be denoted by T and assume that $0 \notin T$. First let us note that if f is any other idempotent in T, f belongs to B and therefore $e = ef = fe = f$ since T is commutative. It follows from this fact that T is a group since it contains at most one idempotent which acts as an identity.

(i) *To show T is one-dimensional.* If T were two-dimensional, then T would contain a two-cell, [**1**, p. 239], and would, therefore, be both opened and closed in S which is a contradiction. Hence T is a one-dimensional group.

(ii) *To show T is a circle.* First let us note that for any $x \neq 0$ in S^0 there exists $f \in B$ such that $x = xf = fx$. This follows from the fact that for any sequence $s(x)$ of 2^nth roots of x there is an f in $Z(s(x))$ which acts as an identity for $s(x)$. From this fact it follows that e is a right unit for S, since for any x in S, $xe = (xf)e = x(fe) = xf = x$, where f is as above. Thus T can be considered as a compact group acting on S. Hence T is locally connected, since T is a one-dimensional orbit [**1**, p. 248]. Thus T is a circle group, since the only connected, one-dimensional groups are the circle group and the solenoid, and the latter is not locally connected.

(iii) Since T is a circle, $S \backslash T = P \cup Q$, where P and Q are open, disjoint, nonempty sets. $P^* \backslash P^0 = T = Q^* \backslash Q^0$, hence P^* and Q^* satisfy the hypothesis for the corollary of Lemma 5, since $P \subseteq J(e)$ and $Q \subseteq J(e)$. Thus $0 \in P \cap Q$, which is a contradiction. From this last contradiction it follows that 0 belongs to T.

LEMMA 7. *For all real t*, $0 < t < \infty$, $\lim \tau(t)^n = 0$.

PROOF. In the following $\Gamma(a) = \{a, a^2, a^3, \cdots\}^*$. Since the only idempotents in T are 0 and e, there exists at least one t_0 such that $0 \in \Gamma(\tau(t_0))$ for otherwise T would be a group which is a contradiction. Now for $a > t_0$, $a = t_0 + b$ for some real b, hence $\tau(a) = \tau(t_0)\tau(b)$. Thus $\tau(a)^n = \tau(t_0)^n \tau(b)^n$ for all positive integers n. Since $0 \in \Gamma(\tau(t_0))$, $\lim \tau(t_0)^n = 0$. Now either $e \in \Gamma(\tau(b))$ or $0 \in \Gamma(\tau(b))$, but in either case, there is a subsequence $(\tau(b))^n_j$; converging to an idempotent. From this it follows that $\lim \tau(a)^{n_j} = \lim \tau(t_0)^{n_j} \lim \tau(b)^{n_j} = 0 \lim \tau(b)^{n_j} = 0$.

Therefore $\lim \tau(a)^n = 0$ for all real $a > t_0$. For $0 < b < t_0$, there exists an integer n such that $t_0 \leqq nb$, or $t_0/n \leqq b$. Since $\tau(t_0/n)^n = \tau(nt_0/n) = \tau(t_0)$, we have $\Gamma(\tau(t_0)) \subseteq \Gamma(\tau(t_0/n))$ and therefore $0 \in \Gamma(\tau(t_0/n))$. Since $t_0/n \leqq b$, the preceding argument applies, therefore $\lim \tau(b)^n = 0$. Hence the lemma is true.

LEMMA 8. $T = \tau[0, \infty) \cup 0$.

PROOF. The proof of this lemma follows immediately from the proof in [**3**, p. 18] for a more general situation.

LEMMA 9. *T is an I-semigroup and there exists t_0, either finite or equal to ∞ such that $T = \tau[0, t_0)^*$ and τ is one-one on $[0, t_0)$.*

PROOF. Clearly T is an arc with endpoints e and 0. Since e acts as an identity for T and 0 functions as a zero, T is an I-semigroup. If there exists a finite $t_0 > 0$ such that $\tau(t) = 0$, let $t_0 = \inf \{t: \tau(t) = 0\}$. If there exists no t such that $\tau(t) = 0$, let $t_0 = \infty$.

If τ is not one-one on $[0, t_0)$, there must exist real numbers r and s in $[0, t_0)$ such that $r < s$ and $\tau(r) = \tau(s)$. Under this assumption, by a straightforward argument, making use of the fact that for any positive integer n, $\tau(ns - (n-1)r) = \tau(r)$, it can be shown that $T \subseteq \tau[0, s]$, a contradiction since $s < t_0$.

LEMMA 10. *For s in S, $s = fj$ where $f \in B$ and $j \in T$. Moreover, if $fj = hk$ for some f, $h \in B$ and j, $k \in T$, then $j = k$.*

PROOF. Let $\theta: B \times T \rightarrow S$ be defined by $\theta(g, t) = gt$. If θ is not onto there exists p in S such that $p \notin \theta(B \times T)$. Let $\delta: S \backslash p \rightarrow B$ be such that δ is continuous and for x in B, $\delta(x) = x$. Now define $\alpha: B \times T \rightarrow B$ by $\alpha(f, t) = \delta(\theta(f, t))$. Since T is compact and connected it follows that the identity function on B is homotopic to a constant function. For $\alpha(x, e) = \delta(\theta(x, e)) = \delta(xe) = \delta(x) = x$ for any x in B, and $\alpha(x, 0) = \delta(0) = h_0$ for some h_0 in B. Hence i, the identity on B and g the function defined by $g(x) = h_0$ for x in B are homotopic, a contradiction. Thus θ is onto and the first part of the lemma is true.

If $fj = hk$ for f and h in B, j and k in T, then $e(fj) = e(hk)$. Since e is the identity for T, it follows that $j = ej = (ef)j = e(fj) = e(hk) = (eh)k = ek = k$.

LEMMA 11. *For f and h in B, j and k in T, $(fj)(hk) = f(jk)$.*

PROOF. This follows easily from the fact that e is the identity for T and $eg = e$ for any g in B. For $(fj)(hk) = f(je)(hk) = (fj)(eh)k = (fj)(ek) = f(jk)$.

With this lemma the proof of Theorem A is concluded.

EXAMPLES. (1) Let S be the two-cell and define $m(x, y)=x|y|$, where m is the multiplication on S and the right side of the equality denotes multiplication of x by the absolute value of y in the ordinary way. In this example, the representation of $x=ft$, for f in B and t in T is unique. The following example shows that this need not always be true.

(2) Let $[e, f]$, $e\neq f$, be a line segment with multiplication for x and y in $[e, f]$ defined by $xy=x$. Let $L=[e, f]\times[0, 1]$, and for (x, t) and (y, s) let the product $(x, t)(y, s)=(x, ts)$ where the product ts is real multiplication in the unit interval.

In L $[e, f]\times 0$ is a closed ideal, so denote by Q the Rees quotient of L with this ideal and let π be the natural map from L onto Q. From [7] Q is a semigroup. Choose a in $(0, 1)$ and consider a fixed in the following discussion. Define θ: $Q\rightarrow Q$ by

$$\theta(\pi(h, t)) = \begin{cases} \pi(e, t) \text{ for } h = e \text{ or } h = f \text{ and } t \in [0, a], \\ \pi(h, t) \text{ otherwise.} \end{cases}$$

Let $Q_0=\theta(Q)$. By defining multiplication in Q_0 in the obvious way, Q_0 is a semigroup.

Now let $L_1=[e_1, f_1]\times[0, 1]$ be another semigroup as defined above, assuming for x and y in $[e_1, f_1]$, $xy=x$. The set $[e_1, f_1]\times[0, a]$ is a closed ideal of L_1. Denote by P the Rees quotient of L_1 with this closed ideal and let π_1 be the natural map from L_1 to P.

Define δ: $P\cup Q_0\rightarrow P\cup Q_0$ in the following way:

$$\delta(s) = \begin{cases} \pi(e, t) \text{ for } s = \pi_1(e_1, t) \text{ where } t \in (a, 1], \\ \pi(f, t) \text{ for } s = \pi_1(f_1, t) \text{ where } t \in (a, 1], \\ \pi(e, a) \text{ for } s = \pi_1(e_1, t) \text{ where } t \in [0, a], \\ \pi(e, a) \text{ for } s = \pi_1(f_1, t) \text{ where } t \in [0, a], \\ s \text{ otherwise.} \end{cases}$$

Let $S=\delta(P\cup Q_0)$. Clearly S is topologically the two-cell. To define multiplication in S it is necessary to consider only an element from $\delta(P\backslash\pi_1(h\times[0, 1]))$, where $h=e_1$ or f_1, and an element from Q_0. This is true since any pair of elements from other sets in S already have their product defined. For $p=\pi_1(h, t)$ for some $h\in(e_1, f_1)$ and $s=\theta(\pi(g, r))$ for $g\in[e, f]$, r and t in $[0, 1]$, let $sp=\pi(g, rt)$ and $ps=\pi_1(h, rt)$. It is easily seen that multiplication defined in this way is continuous, associative and that for x and y in the boundary $xy=x$.

For the particular element $p=\pi(e, a)$, p can be written as

$p=(\pi(e,\ 1))(\pi(e,\ a))$ and also as $p=(\pi(f,\ 1))(\pi(e,\ a))$ and the idempotents $\pi(e,\ 1)$ and $\pi(f,\ 1)$ are distinct elements. In fact, for any idempotent in $[e_1, f_1]$, $p=(\pi_1(h,\ 1))(\pi(e,\ a))$.

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