TWO THEOREMS CONCERNING CONVERGENCE OF FOURIER SERIES

SHIN-ICHI IZUMI¹

This paper contains two theorems concerning the convergence of Fourier series at a point. The theorems are independent of each other.

1. The Lebesgue convergence criterion reads as follows [5]:

THEOREM 1. *If we put $\phi_x(t) = f(x+t) + f(x-t) - 2s$ and suppose that*

$$(1) \quad \frac{1}{h} \int_0^h \phi_x(t) dt \rightarrow 0 \quad \text{as } h \rightarrow 0$$

and

$$(2) \quad \int_h^\pi \frac{|\phi_x(t) - \phi_x(t+h)|}{t} dt \rightarrow 0 \quad \text{as } h \rightarrow 0,$$

then the Fourier series of $f(t)$ converges to s at $t = x$.

The first condition (1) was generalized by many writers. We shall give here another generalization of (1), namely

THEOREM 2. *If*

$$(3) \quad \frac{1}{h} \int_0^h (\phi_x(t) - \phi_x(t+h)) dt \rightarrow 0 \quad \text{as } h \rightarrow 0$$

and (2) holds, then the Fourier series of $f(t)$ converges at $t = x$.

Evidently (1) implies (3). Conditions (2) and (3) are independent of s and then $\phi_x(t)$ may be replaced by $f(x+t) + f(x-t)$. That is, Theorem 2 is a "convergence criterion" but not a "criterion for convergence to s ." Of course, Theorem 1 may be written as a criterion of the former type but there are many criteria of the latter type.

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If we write

$$\Phi(t) = \int_0^t \phi_x(u) du$$

and

$$\Delta_h \Phi(x) = \Phi(x+h) - \Phi(x),$$

$$\Delta_h^2 \Phi(x) = \Phi(x+2h) - 2\Phi(x+h) + \Phi(x),$$

then (3) becomes

$$(4) \quad h^{-1} \Delta_h^2 \Phi(0) \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

This means that $\Phi(x)$ is smooth at $x=0$. On the other hand condition (1) may be written as

$$(5) \quad h^{-1} \Phi(h) = h^{-1} \Delta_h \Phi(0) \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

This means that $\Phi(t)$ is differentiable to zero at $t=0$.

Theorem 2 follows from

THEOREM 3. *If $\int_{-\pi}^{\pi} \phi_x(t) dt = 0$, then conditions (2) and (3) imply (1).*

Therefore Theorems 1 and 2 are equivalent, even though (3) (or (4)) is more general than (1) (or (5)).

We shall now prove Theorem 3. Let $L_n(t)$ be an even function with period 2π and let

$$\begin{aligned} L_n(t) &= 1/2\pi && \text{on } (0, 2\pi/n), \\ L_n(t) &= \pi/nt^2 && \text{on } (2\pi/n, \pi) \quad (n > 2). \end{aligned}$$

Then $\int_{-\pi}^{\pi} L_n(t) dt = 1$. We set $\phi_x(t) = \phi(t)$ and $h = \pi/n$ for the sake of simplicity and let

$$\tau_n(t) = \int_{-\pi}^{\pi} L_n(u) \phi(t+u) du.$$

We write

$$\begin{aligned} \phi(t) &= [\phi(t) - \tau_n(t)] + \tau_n(t), \\ \int_0^h \phi(t) dt &= \int_0^h [\phi(t) - \tau_n(t)] dt + \int_0^h \tau_n(t) dt = I + J. \end{aligned}$$

Let us show that $I = o(h)$ and $J = o(h)$ as $h \rightarrow 0$.

Now we write

$$\begin{aligned}
I &= \int_0^h dt \int_{-\pi}^{\pi} L_n(u) [\phi(t) - \phi(t+u)] du \\
&= \int_0^h dt \int_0^{\pi} du + \int_0^h dt \int_{-\pi}^0 du \\
&= \int_0^h dt \int_0^{\pi} L_n(u) [2\phi(t) - \phi(t+u) - \phi(t-u)] du \\
&= \int_0^h dt \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} L_n(u) [2\phi(t) - \phi(t+u) - \phi(t-u)] du \\
&= \int_0^h dt \sum_{k=0}^{n-1} \int_0^h L_n(u+kh) [2\phi(t) - \phi(t+u+kh) - \phi(t-u-kh)] du \\
&= \int_0^h dt \int_0^h L_n(u) [2\phi(t) - \phi(t+u) - \phi(t-u)] du \\
&\quad + \sum_{k=1}^{n-1} \int_0^h L_n(u+kh) du \int_0^h [2\phi(t) - \phi(t+u+kh) \\
&\quad \quad \quad - \phi(t-u-kh)] du \\
&= I_1 + I_2,
\end{aligned}$$

say, where

$$\begin{aligned}
I_1 &= \frac{1}{2\pi} \int_0^h du \int_0^h [2\phi(t) - \phi(t+u) - \phi(t-u)] dt \\
&= -\frac{1}{2\pi} \int_0^h [\Phi(h+u) - 2\Phi(h) - \Phi(h-u)] du \\
&= -\frac{1}{2\pi} \int_0^h \Delta_u^2 \Phi(h-u) du = o(h)
\end{aligned}$$

by continuity of $\Phi(x)$ and

$$\begin{aligned}
I_2 &= \sum_{k=1}^{n-1} \int_0^h L_n(u+kh) du \int_0^h [\phi(t) - \phi(t+u+kh)] dt \\
&\quad + \sum_{k=1}^{n-1} \int_0^h L_n(u+kh) du \int_0^h [\phi(t) - \phi(t-u-kh)] dt \\
&= I_{21} + I_{22},
\end{aligned}$$

$$\begin{aligned}
I_{21} &= \sum_{k=1}^{n-1} \int_0^h L_n(u + kh) du \int_0^h \{ [\phi(t) - \phi(t + h)] \\
&\quad + [\phi(t + h) - \phi(t + 2h)] \\
&\quad + \cdots + [\phi(t + (k-1)h) - \phi(t + kh)] \\
&\quad + [\phi(t + kh) - \phi(t + kh + u)] \} dt \\
&= \sum_{k=1}^{n-1} \int_0^h L_n(u + kh) du \left\{ \int_0^{kh} [\phi(t) - \phi(t + h)] dt \right. \\
&\quad \left. + \int_{kh}^{(k+1)h} [\phi(t) - \phi(t + u)] dt \right\} \\
&= I_{211} + I_{212}, \\
I_{22} &= \sum_{k=1}^{n-1} \int_0^h L_n(u + kh) du \int_0^h \{ [\phi(t) - \phi(t - h)] \\
&\quad + [\phi(t - h) - \phi(t - 2h)] \\
&\quad + \cdots + [\phi(t - (k-1)h) - \phi(t - kh)] \\
&\quad + [\phi(t - kh) - \phi(t - kh - u)] \} dt \\
&= \sum_{k=1}^{n-1} \int_0^h L_n(u + kh) du \left\{ \int_0^{(k-1)h} [\phi(t) - \phi(t + h)] dt \right. \\
&\quad \left. + \int_{(k-1)h}^{kh} [\phi(t) - \phi(t + u)] dt \right\} \\
&= I_{221} + I_{222},
\end{aligned}$$

since

$$\begin{aligned}
&\int_0^h [\phi(t) - \phi(t - h)] dt = 0, \\
&\int_0^h [\phi(t - jh) - \phi(t - (j+1)h)] dt \\
&\quad = \int_0^h [\phi(t + (j-1)h) - \phi(t + jh)] dt
\end{aligned}$$

and

$$\int_0^h [\phi(t - kh) - \phi(t - kh - u)] dt = \int_{(k-1)h}^{kh} [\phi(t) - \phi(t + u)] dt.$$

We have

$$\begin{aligned}
I_{211} &= \sum_{k=1}^{n-1} \int_0^h L_n(u + kh) du \sum_{j=0}^{k-1} \int_{jh}^{(j+1)h} [\phi(t) - \phi(t+h)] dt \\
&= \sum_{k=1}^{n-1} \int_0^h L_n(u + kh) du \int_0^h [\phi(t) - \phi(t+h)] dt + \sum_{k=1}^{n-1} \sum_{j=1}^{k-1} \\
&= I'_{211} + I''_{211},
\end{aligned}$$

where

$$I'_{211} = \int_h^\pi L_n(u) du \int_0^h [\phi(t) - \phi(t+h)] dt.$$

Similarly putting $I_{221} = I'_{221} + I''_{221}$, we have

$$\begin{aligned}
I'_{211} + I'_{221} &= \left\{ \frac{1}{2} \left(1 - \frac{1}{n} \right) + \frac{1}{2} \left(1 - \frac{2}{n} \right) \right\} \int_0^h [\phi(t) - \phi(t+h)] dt \\
&= \left(1 - \frac{3}{2n} \right) \Delta_h^2 \Phi(0) = o(h)
\end{aligned}$$

by (3). Further

$$\begin{aligned}
I''_{211} &= \sum_{k=1}^{n-1} \sum_{j=1}^{k-1} = \sum_{j=1}^{n-2} \int_{jh}^{(j+1)h} [\phi(t) - \phi(t+h)] dt \sum_{k=j+1}^{n-1} \int_0^h L_n(u + kh) du \\
&= \sum_{j=1}^{n-2} \int_{jh}^{(j+1)h} [\phi(t) - \phi(t+h)] dt \int_{(j+1)h}^\pi L_n(u) du, \\
|I''_{211}| &\leq \frac{\pi}{n} \sum_{j=1}^{n-2} \frac{1}{(j+1)h} \left| \int_{jh}^{(j+1)h} [\phi(t) - \phi(t+h)] dt \right| \\
&\leq 2h \int_h^\pi \frac{|\phi(t) - \phi(t+h)|}{t} dt = o(h)
\end{aligned}$$

by (2).

A similar estimate holds for I''_{221} . We have

$$I_{212} = \sum_{k=1}^{n-1} \int_0^h L_n(u + kh) du \int_{kh}^{(k+1)h} [\phi(t) - \phi(t+u)] dt$$

and then

$$\begin{aligned}
|I_{212}| &\leq \frac{\pi}{n} \sum_{k=1}^{n-1} \int_0^h \frac{kh}{(u+kh)^2} du \int_{kh}^{(k+1)h} \frac{|\phi(t) - \phi(t+u)|}{t} dt \\
&\leq A \int_0^h du \sum_{k=1}^{n-1} \frac{1}{k} \int_{kh}^{(k+1)h} \frac{|\phi(t) - \phi(t+u)|}{t} dt
\end{aligned}$$

which is $o(h)$ by Abel's lemma and condition (2). A similar estimate holds for I_{222} .

It remains to estimate J . Let us put $L_n^*(t) = L_n(t) - 1/2\pi$; then

$$\int_u^\pi L_n^*(t) dt = 0 \quad (0 \leq u \leq 2h)$$

since $L_n^*(t) = 0$ for $0 \leq t \leq 2h$. By $\int_{-\pi}^\pi \phi(u) du = 0$, we get

$$\begin{aligned} J &= \int_0^h \tau_n(t) dt = \int_0^h dt \int_{-\pi}^\pi \phi(u) L_n^*(u-t) du \\ &= \int_{-\pi}^\pi \phi(u) du \int_0^h L_n^*(u-t) dt \\ &= \int_{-\pi}^\pi \phi(u) du \int_{u-h}^u L_n^*(t) dt = \int_{-\pi}^\pi \phi(u) du \left(\int_{u-h}^\pi - \int_u^\pi \right) L_n^*(t) dt \\ &= \int_{-\pi}^\pi \phi(u+h) du \int_u^\pi L_n^*(t) dt - \int_{-\pi}^\pi \phi(u) du \int_u^\pi L_n^*(t) dt, \\ |J| &\leq \int_{-\pi}^\pi |\phi(u+h) - \phi(u)| du \left| \int_{|u|}^\pi L_n^*(t) dt \right| \end{aligned}$$

and then

$$|J| \leq Ah \int_h^\pi \frac{|\phi(u) - \phi(u+h)|}{u} du = o(h).$$

Thus the theorem is proved.

There is an example² of a function which satisfies condition (1) but not (2). Putting $\phi_1(u) = u^{-1}\Phi(u)$, it is sufficient to find a function $\phi_1(u)$ such that (i) $u\phi_1(u)$ is absolutely continuous, (ii) $\phi_1(2u) - \phi_1(u) \rightarrow 0$ as $u \rightarrow 0$, but (iii) $\phi_1(u)$ does not converge as $u \rightarrow 0$.

Let $\phi_1(u)$ be a continuous and differentiable function in the interval $(\pi/2, \pi)$ such that the differential coefficient is bounded and $\phi_1(u)$ takes values 0 and 1. We put

$$\phi_1(u) = \phi_1(2^k u) \text{ for } (\pi/2^{k+1}, \pi/2^k), \quad k = 1, 2, 3, \dots$$

Then $\phi_1(u)$ is defined in the interval $(0, \pi)$. Evidently

$$\phi_1(2u) - \phi_1(u) = 0 \text{ in } (0, \pi/2)$$

and then (ii) is satisfied, but $\phi_1(u)$ takes values 0 and 1 in the interval $(\pi/2^{k+1}, \pi/2^k)$, $k = 1, 2, 3, \dots$, so that (iii) is also satisfied. Now

² This example is due to Professor Kakutani, to whom the author expresses his hearty thanks.

$$(u\phi_1(u))' = \phi_1(u) + u\phi_1'(u)$$

where $\phi_1(u)$ is bounded and then integrable. For u in the interval $(\pi/2^{k+1}, \pi/2^k)$,

$$u\phi_1'(u) = u(\phi_1(2^k u))' = 2^k u\phi_1'(2^k u) = v\phi_1'(v)$$

where $v = 2^k u$ lies in the interval $(\pi/2, \pi)$, and then $u\phi_1'(u)$ is bounded in the interval $(0, \pi)$ and hence is integrable. Thus condition (i) is satisfied.

Finally we remark that Theorem 3 belongs to the same category as the Hille-Klein theorem [1] and its proof is similar to theirs (cf. [2]).

2. M. E. Noble [4] proved the following theorem which contains a well known convergence criterion of Hardy and Littlewood and a lacunary analogue suggested by R. P. Boas.

THEOREM 1. *Suppose that $f(t)$ is an integrable periodic function and*

$$(1) \quad \int_0^t |f(x+u) - f(x)| du = o(t\phi(t))$$

where $\phi(t) = O(1/\log(1/t))$ as $t \downarrow 0$ and is an increasing continuous function of t when $t \geq 0$ with $\phi(0) = 0$. Suppose further that there are sequences of integers (n_k) and (λ_n) such that

$$(2) \quad \int_{1/n_k}^{1/\lambda_{n_k}} \frac{\phi(u)}{u} du = O(1)$$

and

$$(3) \quad \liminf_{k \rightarrow \infty} \min_{n_k - \lambda_{n_k} < n < m < n_k + \lambda_{n_k}} [s_m(x; f) - s_n(x; f)] \geq 0.$$

Then $s_{n_k}(x; f) \rightarrow f(x)$ as $k \rightarrow \infty$.

Noble raised the problem: whether condition (2) could be weakened to

$$(2') \quad \limsup_{k \rightarrow \infty} \int_{1/n_k}^{1/\lambda_{n_k}} \frac{\phi(u)}{u} du = +\infty$$

and he proved that it cannot when there is a constant $\lambda > 0$ such that

$$(4) \quad \liminf_{t \downarrow 0} \{ \phi(kt)/\phi(t) \} \geq \lambda$$

for any k , $0 < k < 1$.

Here we shall prove

THEOREM 2. *Suppose that $\phi(t)$ is an increasing continuous function of t when $t \geq 0$ with $\phi(0) = 0$ and that (λ_n) is a sequence of positive numbers such that $\lambda_n = o(n)$ and*

$$\limsup_{n \rightarrow \infty} \int_{1/n}^{1/\lambda_n} \frac{\phi(t)}{t} dt = \infty.$$

Then there are a continuous function $f(x)$ and a sequence (n_k) such that

$$(5) \quad f(t) = o(\phi(t)) \text{ as } t \rightarrow 0,$$

$$(6) \quad \lim_{k \rightarrow \infty} \max_{n_k < \lambda_{n_k} < n < m < \lambda_{n_k} + \lambda_{n_k}} |s_m(0; f) - s_n(0; f)| = 0$$

and

$$(7) \quad s_{n_k}(0; f) \rightarrow \infty.$$

PROOF.³ We can suppose that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $f(t)$ be an even function such that

$$\begin{aligned} f(t) &= c_k \phi(t) \sin n_k t & \text{for } t \in I_k = (1/n_k, 1/\lambda_{n_k}) \\ &= 0 & \text{otherwise} \end{aligned}$$

where (n_k) is taken such that the intervals I_k do not overlap and

$$(8) \quad \int_{I_k} \frac{\phi(t)}{t} dt \rightarrow \infty \quad \text{as } k \rightarrow \infty$$

and (c_k) is a sequence of positive numbers. The sequence (n_k) and (c_k) will be determined later. We have

$$|f(t)| \leq c_k \phi(t) \quad \text{for } t \in (1/n_k, 1/n_{k-1})$$

and then (5) holds if $c_k \rightarrow 0$. Now

$$\begin{aligned} s_{n_k}(0; f) &= \frac{2}{\pi} \int_0^\pi f(t) \frac{\sin n_k t}{t} dt + o(1) \\ &= \frac{2}{\pi} \sum_{j=1}^\infty c_j \int_{I_j} \phi(t) \sin n_j t \frac{\sin n_k t}{t} dt + o(1) = \frac{2}{\pi} \sum_{j=1}^\infty d_j + o(1). \end{aligned}$$

We have

$$d_k = c_k \left\{ \frac{1}{2} \int_{I_k} \frac{\phi(t)}{t} dt - \frac{1}{2} \int_{I_k} \frac{\phi(t)}{t} \cos 2n_k t dt \right\}$$

where

³ Cf. [3].

$$\left| \int_{I_k} \frac{\phi(t)}{t} \cos 2n_k t dt \right| \leq n_k \phi(1/\lambda_{n_k}) \left| \int_{I'_k} \cos 2n_k t dt \right| = o(1) \quad (I'_k \subset I_k).$$

Therefore we get

$$(9) \quad d_k \geq A c_k \int_{I_k} \frac{\phi(t)}{t} dt$$

which tends to infinity by (8), if c_k tends sufficiently slowly to zero.

Now, for $j < k$, if the sequence (n_k) has the Hadamard gap condition, then

$$\begin{aligned} d_j &= c_j \int_{I_j} \frac{\phi(t)}{t} \sin n_j t \sin n_k t dt \\ &= \frac{1}{2} c_j \int_{I_j} \frac{\phi(t)}{t} (\cos (n_k - n_j)t - \cos (n_k + n_j)t) dt, \\ |d_j| &\leq \frac{1}{2} c_j n_j \phi(1/\lambda_{n_j}) \left| \int_{I_j''} (\cos (n_k - n_j)t - \cos (n_k + n_j)t) dt \right| \\ &\leq A c_j n_j / n_k \end{aligned}$$

where $I_j'' \subset I_j$, and hence

$$(10) \quad \sum_{j=1}^{k-1} |d_j| \leq A \sum_{j=1}^{k-1} c_j n_j / n_k = o(1),$$

if (n_k) has Hadamard gaps.

For $j > k$, we have $|d_j| \leq A \phi(1/\lambda_{n_j})$. If the sequence (n_k) increases sufficiently rapidly so that $\sum \phi(1/\lambda_{n_j}) < \infty$, then

$$(11) \quad \sum_{j=k+1}^{\infty} |d_j| = o(1).$$

Equation (7) follows from (9), (10) and (11).

It remains to prove (6). Let $n_k - \lambda_{n_k} < n < m < n_k + \lambda_{n_k}$. Then

$$\begin{aligned} \frac{\pi}{2} (s_m(0; f) - s_n(0; f)) &= \int_0^{\pi} f(t) \frac{\sin mt - \sin nt}{t} dt + o(1) \\ &= \sum_{j=1}^{\infty} c_j \int_{I_j} \phi(t) \sin n_j t \frac{\sin mt - \sin nt}{t} dt + o(1) \\ (12) \quad &= 2 \sum_{j=1}^{\infty} c_j \int_{I_j} \frac{\phi(t)}{t} \sin n_j t \sin \frac{1}{2} (m - n)t \cos \frac{1}{2} (m + n)t dt + o(1) \\ &= 2 \sum_{j=1}^{\infty} e_j + o(1). \end{aligned}$$

We have

$$\begin{aligned} e_j &= c_j \int_{1/n_j}^{1/\lambda_{n_j}} \frac{\phi(t)}{t} \sin n_j t \sin \frac{1}{2}(m-n)t \cos \frac{1}{2}(m+n)t dt, \\ |e_j| &\leq c_j \frac{m-n}{2} \int_{1/n_j}^{1/\lambda_{n_j}} \phi(t) dt \\ &\leq c_j \frac{\lambda_{n_k}}{\lambda_{n_j}} \phi(1/\lambda_{n_j}). \end{aligned}$$

This gives, as for d_j ,

$$\sum_{j=k+1}^{\infty} |e_j| = o(1)$$

when (n_k) has sufficiently large gaps, since $\lambda_k \rightarrow \infty$ and $\phi(t) \rightarrow 0$ as $t \rightarrow 0$.

On the other hand,

$$e_j = c_j \int_{I_j} \frac{\phi(t)}{t} \sin n_j t \sin m t dt - c_j \int_{I_j} \frac{\phi(t)}{t} \sin n_j t \sin n t dt.$$

If $j \leq k-1$, then n_j is less than m and n , so that, if (n_k) has sufficiently large gaps, the estimate for d_j to deduce (10) is now used to get

$$\sum_{j=1}^{k-1} |e_j| = o(1).$$

Finally it remains to prove that $e_k = o(1)$. We can suppose that $n = n_k$ and $m = n_k + \mu_k > n_k$. Then

$$\begin{aligned} e_k &= c_k \int_{I_k} \frac{\phi(t)}{t} \sin n_k t (\sin (n_k + \mu_k)t - \sin n_k t) dt \\ &= 2c_k \int_{I_k} \frac{\phi(t)}{t} \sin n_k t \cos (n_k + \mu_k/2)t \sin (\mu_k/2)t dt \\ &= 2c_k \sin \frac{\mu_k}{2\lambda_{n_k}} \phi(1/\lambda_{n_k}) \int_{1/n'_k}^{1/\lambda_{n_k}} \frac{\sin (2n_k + \mu_k/2)t - \sin \mu_k t/2}{t} dt \end{aligned}$$

where $n_k > n'_k > \lambda_{n_k}$, since $\sin \mu_k t/2$ is monotone in the interval I_k . Therefore $e_k = o(1)$. Thus the theorem is proved.

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UNIVERSITY OF CHICAGO AND
NORTHWESTERN UNIVERSITY

AN EXTREMAL PROBLEM FOR POLYNOMIALS

FRANCIS P. CALLAHAN, JR.

PROBLEM. Consider the class of n th order polynomials $\{f(z)\}$ such that $f(1)=0$, $|f(z)| \leq 1$ for $|z|=1$. From this class select that polynomial for which

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \text{ is greatest.}$$

For the solution we require the following

LEMMA. Let $h(z) = \sum_{-N}^N h_n z^n$, ($\bar{h}_n = h_{-n}$). Then there exists a polynomial $f(z)$ of degree N such that $h(z) = |f(z)|^2$ for $|z|=1$ if and only if $h(z) \geq 0$ for $|z|=1$. Proof is available in reference [1].

The function $1 - |f(z)|^2$ (with $\bar{z}$ replaced by $1/z$) satisfies the conditions of the lemma for any $f(z)$ that satisfies the conditions of the problem. Thus, we can write, $1 - |f(z)|^2 = |g(z)|^2$, where $g(z)$ satisfies the conditions that $|g(z)| \leq 1$ for $|z|=1$ and $|g(1)|=1$. (Without real loss of generality, we take this last to mean $g(1)=1$.)

In addition, for $f(z)$ to solve the problem, the associated $g(z)$ must minimize the integral

$$\frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^2 d\theta.$$

Writing $g(z) = \sum_0^N g_n z^n$, we see that we are seeking to minimize the quantity $\sum_0^N |g_n|^2$ subject to the constraint that $\sum_0^N g_n = 1$. A straightforward application of the Schwarz Inequality yields: $1 = \sum_0^N g_n \leq (\sum_0^N |g_n|^2)^{1/2} (N+1)^{1/2}$. The sum-of-squares is smallest when we set $g_n = 1/(N+1)$, and obtain for the corresponding $g(z)$,

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