AN OSCILLATION THEOREM FOR SOLUTIONS OF A CLASS OF PARTIAL DIFFERENCE EQUATIONS

FULTON KOEHLER AND CHARLES M. BRADEN

In 1908 Hadamard [2] raised the question: if u is the solution of

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = p$$

in a region R with $u = \partial u / \partial n = 0$ on the boundary of R , does the restriction $p \geq 0$ imply that $u \geq 0$? Duffin, [1], showed that it is possible, when R is an infinite strip, to construct solutions for this problem which oscillate in sign infinitely often, where p is non-negative everywhere and zero outside a finite subset of R . Loewner [4] and Szegő [5] have constructed other examples showing this same kind of behavior. In this paper we consider a finite difference analogue of Duffin's problem, but prove results analogous to his for a general class of partial difference equations, of which the biharmonic equation is a special case.

Let $u(x, y)$ satisfy a linear partial difference equation of the form:

$$(1) \quad Lu(x, y) = \sum_{i=-\nu}^{\nu} \sum_{j=-m}^m c_{ij} u(x+i, y+j) = 0$$

for all integral (x, y) in the strip $S_N: 0 \leq y \leq N, x \geq 0$. The boundary values of u for $-m \leq y < 0$ and for $N < y \leq N+m$ are assumed to be zero, and those for $-\nu \leq x < 0, 0 \leq y \leq N$ are arbitrary real numbers. The coefficients c_{ij} are assumed to be real constants, and $c_{ij} = c_{i,-j}$. Let

$$\begin{vmatrix} c_{\nu,0} & c_{\nu,1} & \cdots & c_{\nu,N} \\ c_{\nu,-1} & c_{\nu,0} & \cdots & c_{\nu,N-1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{\nu,-N} & c_{\nu,-N+1} & \cdots & c_{\nu,0} \end{vmatrix} \neq 0,$$

where $c_{ij} = 0$ if $|i| > \nu$ or $|j| > m$. Then the solution $u(x, y)$ is uniquely determined by its boundary values and by its values for $0 \leq x \leq \nu-1, 0 \leq y \leq N$. These values are assumed to be real, and $u(x, y)$ is then real for all (x, y) in the strip S_N .

Let $Eu(x, y) = u(x+1, y)$, $Q_j(E) = \sum_{i=-\nu}^{\nu} c_{ij} E^i$, and let the partial

Received by the editors January 8, 1959.

difference equation (1) be thought of as a system of ordinary difference equations in $u(x, 0), u(x, 1), \dots, u(x, N)$. We then have, for each integer y in $[0, N]$,

$$\Delta(E)u(x, y) = 0, \quad x \geq 0,$$

where

$$(2) \quad \Delta(E) = \begin{vmatrix} Q_0(E) & Q_1(E) & \cdots & Q_N(E) \\ Q_{-1}(E) & Q_0(E) & \cdots & Q_{N-1}(E) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ Q_{-N}(E) & Q_{-N+1}(E) & \cdots & Q_0(E) \end{vmatrix}.$$

Let the distinct roots of the equation $\Delta(z) = 0$ be $z = r_1, r_2, \dots, r_K$, and let these roots have multiplicities $1 + s_1, 1 + s_2, \dots, 1 + s_K$ respectively. We shall refer to these roots as the characteristic roots of the operator L for the strip S_N . Then

$$(3) \quad u(x, y) = \sum_{\rho=1}^K \sum_{\sigma=0}^{s_\rho} a_{\rho\sigma} x^\sigma r_\rho^x,$$

where the coefficients $a_{\rho\sigma}$ are functions of y .

We shall be particularly concerned with operators L whose characteristic roots are all nonreal and take up first a sufficient condition for an operator to have this property. Let

$$(4) \quad f(r, \theta) = \sum_{j=-N}^N Q_j(r) e^{ij\theta} = Q_0(r) + 2 \sum_{j=1}^N Q_j(r) \cos j\theta.$$

THEOREM 1. *If, for every real number r , $f(r, \theta)$ is non-negative and not identically zero for $0 \leq \theta \leq 2\pi$, then the characteristic roots of the operator L are all nonreal.*

PROOF. Let $D(r, \xi)$ be the Toeplitz form corresponding to $f(r, \theta)$,

$$(5) \quad D(r, \xi) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) |T(\xi, \theta)|^2 d\theta = \sum_{i=0}^N \sum_{j=0}^N Q_{i-j}(r) \xi_i \xi_j,$$

where $T(\xi, \theta) = \sum_{k=0}^N \xi_k e^{ik\theta}$, ξ_k real. Then, for any real r , this quadratic form is positive definite, and therefore its determinant cannot be zero, i.e.: $\Delta(r) \neq 0$.

THEOREM 2. *Let L_1 and L_2 be two operators of the form defined by (1), and let $f_1(r, \theta)$ and $f_2(r, \theta)$ be the corresponding functions defined by (4). Then the function defined by (4) for the operator $L_1 L_2$ is $f_1 f_2$, provided N is sufficiently large.*

PROOF. Let

$$L_1(u, y) = \sum_{j=-m_1}^{m_1} Q_j^{(1)}(E)u(x, y + j),$$

$$L_2u(x, y) = \sum_{j=-m_2}^{m_2} Q_j^{(2)}(E)u(x, y + j),$$

and let $Q_j^{(1)} = 0$ for $|j| > m_1$, $Q_j^{(2)} = 0$ for $|j| > m_2$. Then

$$\begin{aligned} L_1L_2u(x, y) &= \sum_{j=-m_1}^{m_1} Q_j^{(1)}(E) \sum_{k=-m_2}^{m_2} Q_k^{(2)}(E)u(x, y + k + j) \\ &= \sum_{l=-(m_1+m_2)}^{m_1+m_2} Q_l(E)u(x, y + l), \end{aligned}$$

where $Q_l(E) = \sum_{j+k=l} Q_j^{(1)}Q_k^{(2)}$. Hence, if $N \geq m_1 + m_2$, the function $f(r, \theta)$ defined by (4) for the operator L_1L_2 is given by

$$f(r, \theta) = \sum_{l=-\infty}^{\infty} Q_l(r)e^{il\theta} = \sum_{l=-\infty}^{\infty} \sum_{j+k=l} Q_j^{(1)}Q_k^{(2)}e^{il\theta} = f_1f_2.$$

The usual difference operator, $L^{(H)}$, arising from Laplace's equation is defined by

$$L^{(H)}u(x, y) = \sum_{j=-1}^1 Q_j^{(H)}(E)u(x, y + j),$$

where $Q_0^{(H)} = E^{-1} - 4 + E$, $Q_{\pm 1}^{(H)} = Q_{\mp 1}^{(H)} = 1$. The usual operator $L^{(B)}$ arising from the biharmonic equation is taken equal to $L^{(H)^2}$, and

$$L^{(B)}u(x, y) = \sum_{j=-2}^2 Q_j^{(B)}(E)u(x, y + j),$$

where $Q_0^{(B)} = E^{-2} - 8E^{-1} + 20 - 8E + E^2$, $Q_1^{(B)} = Q_{-1}^{(B)} = 2E^{-1} - 8 + 2E$, $Q_{\pm 2}^{(B)} = 1$.

THEOREM 3. *The characteristic roots of the biharmonic operator are nonreal for all values of N .*

PROOF. For $N \geq 2$, the result follows from Theorems 1 and 2 and the fact that the biharmonic operator is the square of the harmonic operator. For $N = 0, 1$ the determinants $\Delta(r)$ defined by (2) become

$$\Delta_0(r) = \rho^2 - 8\rho + 18$$

and

$$\Delta_1(r) = (\rho^2 - 8\rho + 18)^2 - (2\rho - 8)^2 = (\rho^2 - 6\rho + 10)(\rho^2 - 10\rho + 26)$$

respectively, where $\rho = r + 1/r \geq 2$, and it is obvious that $\Delta_0 > 0$, $\Delta_1 > 0$ for real r .

Let us assume that the characteristic roots of an operator L for the strip S_N are all nonreal. For any fixed integer y in $[0, N]$ let any real solution (3) of (1) be written in the form

$$(6) \quad u(x, y) = \sum_{\alpha=1}^l \rho_{\alpha}^x (q_{\alpha} e^{ix\theta_{\alpha}} + \bar{q}_{\alpha} e^{-ix\theta_{\alpha}}),$$

where $\rho_1 \geq \rho_2 \geq \dots \geq \rho_l > 0$, and where, for each α , $0 < \theta_{\alpha} < \pi$, and q_{α} is a polynomial in x not identically zero. Let $k \geq 1$ be the integer such that $\rho_1 = \rho_2 = \dots = \rho_k$ and either $k = l$ or $\rho_k > \rho_{k+1}$. Then

$$u(x, y) = \rho_1^x \left[\sum_{\alpha=1}^k (q_{\alpha} e^{ix\theta_{\alpha}} + \bar{q}_{\alpha} e^{-ix\theta_{\alpha}}) + o(1) \right] \quad \text{as } x \rightarrow \infty,$$

the numbers $\theta_1, \theta_2, \dots, \theta_k$ being distinct. Let μ be the maximum degree of the polynomials q_{α} , $1 \leq \alpha \leq k$. Then

$$(7) \quad u(x, y) = \rho_1^x x^{\mu} \left[\sum_{\alpha=1}^k (c_{\alpha} e^{ix\theta_{\alpha}} + \bar{c}_{\alpha} e^{-ix\theta_{\alpha}}) + o(1) \right],$$

where $c_1, c_2, \dots, c_k$ are constants, not all zero.

THEOREM 4. *If the characteristic roots of the operator L in the strip S_N are all nonreal, then the solution of (1), for each y in $[0, N]$, will change sign infinitely many times as $x \rightarrow \infty$.*

PROOF. For any fixed y in $[0, N]$ let $u(x, y)$ be expressed by (7). Let

$$T(x) = \sum_{\alpha=1}^k (c_{\alpha} e^{ix\theta_{\alpha}} + \bar{c}_{\alpha} e^{-ix\theta_{\alpha}}).$$

For any $\epsilon > 0$ let I_{ϵ} be the set of positive integers ν for which there exist corresponding integers ν_{α} such that

$$|\nu\theta_{\alpha} - 2\pi\nu_{\alpha}| < \epsilon, \quad \alpha = 1, 2, \dots, k.$$

For any $\epsilon > 0$, I_{ϵ} contains infinitely many members, (see [3, p. 169]), and if $\nu \in I_{\epsilon}$,

$$(8) \quad |T(x) - T(x + \nu)| < C\epsilon, \quad C = 2 \sum_{\alpha=1}^k |c_{\alpha}|.$$

For any positive integer A , let us suppose that there are no sign changes in $T(x)$ for $x \geq A$, say $T(x) \geq 0$ for $x \geq A$. Choose $x_0 \geq A$ such that $T(x_0) > 0$, and let $\epsilon = (1/2C)T(x_0)$. Then $T(x) > 2^{-1}T(x_0)$ for

$x = x_0 + \nu$, $\nu \in I_\epsilon$ and $\sum_{x=A}^M T(x) \rightarrow \infty$ as $M \rightarrow \infty$. This, however, is a contradiction, since

$$\left| \sum_{x=A}^M T(x) \right| \leq \frac{2C}{|1 - e^{i\theta}|}, \quad \theta = \min_{\alpha} \{\theta_{\alpha}\},$$

for all M . The assumption $T(x) \leq 0$ for $x \geq A$ also leads to a contradiction. Hence, for any A , $T(x)$ must change sign at least once, and consequently an infinite number of times, for $x \geq A$.

Let $T(x_0) > 0$, $\epsilon = (1/2C)T(x_0)$. Then, from (7), $u(x_0 + \nu, y) > 0$ for all sufficiently large values of ν in I_ϵ . In the same way we show that there must exist infinitely many values of x such that $u(x, y) < 0$.

Returning to Hadamard's question stated in the first paragraph, let us assume that u satisfies the partial difference equation $Lu = p$ in the doubly infinite strip $0 \leq y \leq N$ and that $u = 0$ for $-m \leq y < 0$ and $N < y \leq N + m$. If the characteristic roots of L in the strip S_N are nonreal and if $p(x, y) = 0$ for $x \geq 0$, then, by Theorem 4, $u(x, y)$ oscillates in sign infinitely often as $x \rightarrow \infty$. This is true regardless of the values of $p(x, y)$ for $x < 0$.

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UNIVERSITY OF MINNESOTA AND
MACALESTER COLLEGE