ON LIMITS OF MODULE-SYSTEMS

JOHANN SONNER

1. **Introduction.** This paper is concerned with a generalization of direct and inverse limits. The construction uses local sections defined on sets of a filter $\mathfrak{F}$ on M. Usually, M is a directed set (see [1]). The purpose of the order-relation is a twofold one: First it serves to generate a filter on M, second it is responsible for the existence of certain homomorphisms π_α^β. That both tasks can be separated was already pointed out in [2]; that only the filtering property of the order relation is necessary for the construction of limits will be shown here.

2. **Definition of module-systems.**

DEFINITION 1. A pair (f, g), where f is a surjective map of E into M and g a map of M^2 into $\mathfrak{P}(E^2)$, defines on a pair (E, M) of sets a *structure of a module-system* over the ring R if it possesses the following properties:

(SM_I) For each $\alpha \in M$, the reciprocal image $f^{-1}(\alpha)$ of α under f is a module over R;

(SM_{II}) For each pair $(\alpha, \beta) \in M^2$, the part $g(\alpha, \beta)$ of E^2 is a submodule of the product-module $f^{-1}(\alpha) \times f^{-1}(\beta)$ over R;

(SM_{III}) For each $\alpha \in M$, the set $g(\alpha, \alpha)$ is a part of the diagonal of $f^{-1}(\alpha) \times f^{-1}(\alpha)$.

Write E_α instead of $f^{-1}(\alpha)$.

EXAMPLE. Let E be the sum of the sets E_α ($\alpha \in M$) of a direct (resp. inverse) system of R-modules over the directed set M and f the map of E into M which assigns to each $x \in E_\alpha$ the element $\alpha \in M$. If $\alpha \leqq \beta$, denote with $g(\alpha, \beta)$ (resp. $g(\beta, \alpha)$) the graph of the homomorphism π_α^β, if not $\alpha \leqq \beta$, let $g(\alpha, \beta) = E_\alpha \times E_\beta$. The pair (E, M) together with (f, g) forms an R-module-system.

Definition 1 can readily be modified in order to suit any algebraic structure, e.g. group-system, vector-space-system.

3. **Construction of limit-modules.** Let (E, M) be a module-system and $\mathfrak{F}$ a filter on M. A function $s = (S, A, E)$ defined on a set $A \in \mathfrak{F}$ with values in E is a *local section* of f, if the composition $f \circ s$ is the canonical injection of A into M. The section s is said to be *g-admissible* if $(\alpha, \beta) \in M^2$ implies $(s(\alpha), s(\beta)) \in g(\alpha, \beta)$.

Denote the set of the g-admissible local sections of f, defined on a

Presented to the Society, January 21, 1959; received by the editors February 8, 1959.

set $A \in \mathfrak{F}$, by $S(\mathfrak{F}, f, g)$. The relation $\ll$ for some $A \in \mathfrak{F}$, $s_A = t_A \gg$ is an equivalence relation R_∞ in $S(\mathfrak{F}, f, g)$; write $S_\infty(\mathfrak{F}, f, g)$ for $S(\mathfrak{F}, f, g)/R_\infty$. We shall endow $S_\infty(\mathfrak{F}, f, g)$ with the structure of a module over the ring R.

If $s = (S, A, E)$, $t = (T, B, E)$ are g-admissible local sections of f, define $s + t$ by $\alpha \to s(\alpha) + t(\alpha)(\alpha \in A \cap B)$, $\lambda \cdot s$ by $\alpha \to \lambda \cdot s(\alpha)(\alpha \in A)$, which is possible by axiom ($\mathrm{SM_I}$). Clearly, $s + t$ and $\lambda \cdot s$ are local sections which are g-admissible by axiom ($\mathrm{SM_{II}}$). Note, that addition and multiplication are compatible with R_∞. Therefore it is allowed to pass to the quotient, thus obtaining a structure $\mathfrak{M}_\infty$ of a module over the ring R. We call *limit* of the R-module-system (E, M) with respect to the filter $\mathfrak{F}$ and denote by $\lim_{\mathfrak{F}} (f, g)$ the quotient $S_\infty(\mathfrak{F}, f, g)$, endowed with the structure $\mathfrak{M}_\infty$.

EXAMPLE. In the case of a direct system, let $\mathfrak{F}$ be the filter of the sections of the direct set $M \neq \varnothing$, in the case of an inverse system, take the reduced set $\{M\}$. The module $\lim_{\mathfrak{F}} (f, g)$ is isomorphic to the direct (resp. inverse) limit as was proved by Griffiths (see [2]).

4. Maps.

DEFINITION 2. Let $\varphi = (\Phi, E, E')$ and $\psi = (\Psi, M, M')$ be functions. The pair (φ, ψ) is called a map of the module system (E, M) into the module-system (E', M') over the same ring R, if it satisfies the following conditions:

1°. $\psi \circ f = f' \circ \varphi$;

2°. For each pair $(\alpha, \beta) \in M^2$, $(x, y) \in g(\alpha, \beta)$ implies $(\varphi(x), \varphi(y)) \in g'(\psi(\alpha), \psi(\beta))$;

3°. For each $\alpha \in M$, the map φ_α of E_α into $E'_{\psi(\alpha)}$, induced by φ, is a homomorphism.

The composition of two maps is a map; for bijections φ, ψ the condition $\ll(\varphi, \psi)$ and $(\varphi^{-1}, \psi^{-1})$ are maps$\gg$ is equivalent to the condition $\ll(\varphi, \psi)$ is an isomorphism$\gg$.

PROPOSITION 1. *Let (φ, ψ) be a map of a module-system (E, M) with filter $\mathfrak{F}$ into a module-system (E', M') with filter $\mathfrak{F}'$. If $\mathfrak{F}'$ is finer than the filter on M' generated by the image of $\mathfrak{F}$ under ψ, then the pair (φ, ψ) induces a homomorphism $(\varphi, \psi)_\infty$ of* $\lim_{\mathfrak{F}} (f, g)$ *into* $\lim_{\mathfrak{F}'} (f', g')$.

Let $s = (S, A, E)$ be a g-admissible local section of f. Denote by s' the correspondence $\varphi \circ s \circ \psi_A^{-1}$, where ψ_A is the map of A into $\psi(A)$ induced by ψ. s' is a function (Axiom ($\mathrm{SM_{III}}$)) which is a g'-admissible local section of f'. Furthermore, $s \to s'$ is compatible with the equivalence relations R_∞ and R'_∞, thus inducing a homomorphism $(\varphi, \psi)_\infty$ of $S_\infty(\mathfrak{F}, f, g)$ into $S_\infty(\mathfrak{F}', f', g')$.

5. **Induced structures.** Let (E, M) be a R-module-system and N a part of M. Write E_N for the reciprocal image of N under f, f_N for the subjective map of E_N into N induced by f and g_N for the map of N^2 into $\mathfrak{P}(E_N^2)$ induced by g. Obviously, the pair (f_N, g_N) defines on (E_N, N) a structure of a module-system over R, called the *induced structure.* It is obviously an initial structure in the sense of N. Bourbaki.

THEOREM 1. *Let (E, M) be a module-system, N a part of M, i (resp. j) the canonical injection of E_N (resp. N) into E (resp. M), and $\mathfrak{F}$ a filter on N. If $\mathfrak{F}'$ is the filter on M generated by $\mathfrak{F}$, then the map (i, j) induces an isomorphism $(i, j)_\infty$ of* $\lim_{\mathfrak{F}} (f_N, g_N)$ *into* $\lim_{\mathfrak{F}'} (f, g)$.

By Proposition 1, (i, j) induces a homomorphism which is bijective because of $s' = i \circ s$ (see proof of Proposition 1).

6. **Cofinality.** A pair $(\alpha, \beta) \in M^2$ is called *g-distinguished* if the correspondence $(g(\alpha, \beta), E_\alpha, E_\beta)$ is a function.

DEFINITION 3. A filter $\mathfrak{F}'$ is *cofinal* for a filter $\mathfrak{F}$ relative to (f, g), if $\mathfrak{F}'$ is finer than $\mathfrak{F}$, and if for every set $A \in \mathfrak{F}'$, there exists a set $B \in \mathfrak{F}$ satisfying the following conditions:

(CF_{I}) For every element $\beta \in B$, there exists an element $\alpha \in A \cap B$ such that that (α, β) is g-distinguished;

(CF_{II}) If (α_1, β_1) and (α_2, β_2) are g-distinguished elements of $(A \cap B) \times B$, then there exists an element $\alpha \in A \cap B$ such that (α, α_1) is g-distinguished and the relation

$$g(\alpha_2, \beta_2) \circ g(\alpha, \alpha_2) \subset g(\beta_1, \beta_2) \circ g(\alpha_1, \beta_1) \circ g(\alpha, \alpha_1)$$

holds (see diagram below).

A filterbase $\mathfrak{B}'$ is *cofinal* for $\mathfrak{F}$, if the filter $\mathfrak{F}'$ generated by $\mathfrak{B}'$, is cofinal for $\mathfrak{F}$.

$$\begin{tikzcd}
 & E_{\alpha_1} \arrow[r, "g(\alpha_1, \beta_1)"] & E_{\beta_1} \arrow[d, "g(\beta_1, \beta_2)"] \\
E_\alpha \arrow[ur, "g(\alpha, \alpha_1)"] \arrow[dr, "g(\alpha, \alpha_2)"'] & & \\
 & E_{\alpha_2} \arrow[r, "g(\alpha_2, \beta_2)"] & E_{\beta_2}
\end{tikzcd}$$

THEOREM 2. *Let (E, M) be a module-system, i (resp. j) the identity of E (resp. M) and $\mathfrak{F}$, $\mathfrak{F}'$ filters on M. If $\mathfrak{F}'$ is cofinal for $\mathfrak{F}$, then the map (i, j) induces an isomorphism $(i, j)_\infty$ of* $\lim_{\mathfrak{F}} (f, g)$ *into* $\lim_{\mathfrak{F}'} (f, g)$.

By Proposition 1, (i, j) induces a homomorphism which is bijective because of $s' = s$ (see proof of Proposition 1).

THEOREM 3. *Let* (f, g), (f', g') *be pairs of maps, defining structures of* R*-module-systems on* (E, M), $\mathfrak{F}$, $\mathfrak{F}'$ *filters on* M *and* N *a part of* M. *Suppose, that the induced structures and filters on* N *exist and coincide, and that* $\mathfrak{F}_N$ *(resp.* $\mathfrak{F}'_N$*) is cofinal for* $\mathfrak{F}$ *(resp.* $\mathfrak{F}'$*) relative to* (f, g) *(resp.* (f', g')*). Then the modules* $\lim_{\mathfrak{F}} (f, g)$ *and* $\lim_{\mathfrak{F}'} (f', g')$ *are isomorphic.*

Write $\mathfrak{G}$ (resp. $\mathfrak{G}'$) for the filter on M generated by $\mathfrak{F}_N$ (resp. $\mathfrak{F}'_N$). Consider the diagram

$$\lim_{\mathfrak{F}} (f, g) \longrightarrow \lim_{\mathfrak{G}} (f, g) \longleftarrow \lim_{\mathfrak{F}_N} (f_N, g_N).$$

The first map is induced by a pair of identities and is an isomorphism according to Theorem 2, the second map is induced by a pair of canonical injections and is an isomorphism according to Theorem 1. Because the same argument holds for the primed maps, the theorem follows.

REFERENCES

1. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Mathematical Series, no. 15, 1957.

2. H. B. Griffiths, *On limits of systems of groups*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 118–129.

UNIVERSITY OF SOUTH CAROLINA