

$$(4.3) \quad f(z) = z(1 + z^2)/(1 - z^2)^2$$

is typically-real in the unit circle. Moreover, since the derivative of this function vanishes at $z = \pm((2)^{1/2} - 1)i$, (4.3) is not univalent in any larger circular domain with center at the origin than that given in Corollary 4.1.

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ON THE POLE AND ZERO LOCATIONS OF RATIONAL LAPLACE TRANSFORMATIONS OF NON-NEGATIVE FUNCTIONS

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Let $a(t)$ be a real, bounded function of the real variable t defined in the interval, $0 \leq t < \infty$, and let its Laplace-Stieltjes transform,

$$(1) \quad F(s) = \int_0^\infty e^{-st} da(t),$$

be a rational function of the complex variable $s = \sigma + i\omega$, having at least as many poles as zeros. $F(s)$ may be written as

$$(2) \quad F(s) = \frac{\prod_{i=1}^h (s - \eta_i) \prod_{i=1}^g (s - \nu_i)}{\prod_{i=1}^m (s - \rho_i) \prod_{i=1}^q (s - \xi_i)}$$

where the η_i and ρ_i are real and the ν_i and ξ_i are complex. Under these

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conditions, it follows that $a(t)$ is a sum of exponential functions plus possibly a constant. Moreover, the complex poles and zeros of $F(s)$ will occur in complex conjugate pairs and all the poles will have negative real parts.

The purpose of this note is to present a sufficient set of conditions on the pole and zero locations of $F(s)$ which insure that $a(t)$ is a real, nondecreasing, and bounded function. Mulligan [1], Lukacs and Szász, [2; 3], and Takano [4] have developed necessary conditions on the pole and zero locations for such an $a(t)$. For example, if $\alpha + i\beta$ ($\alpha < 0$) is a pole of $F(s)$, then $F(s)$ has at least one real pole ρ such that $\alpha \leq \rho < 0$. Furthermore, Lukacs and Szász [5; 6] have obtained sufficient conditions by restricting their investigation to the case where all the poles and zeros are simple and have the same real parts.

The method of proof in this paper makes use of a theorem due to Bernstein [7, Theorem 12a, p. 160] about completely monotonic functions. A function $f(x)$ is said to be completely monotonic in the interval $0 \leq x < \infty$ if it is continuous there and has derivatives of all orders satisfying

$$(3) \quad (-1)^k f^{(k)}(x) \geq 0, \quad (0 < x < \infty; k = 0, 1, 2, \dots).$$

BERNSTEIN'S THEOREM. "A necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \leq x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} dg(t)$$

where $g(t)$ is bounded and nondecreasing and the integral converges for $0 \leq x < \infty$."

Now consider a function $F(s)$ of the form (2) where $m \geq n = h + g + q$. The η_i , ν_i , ρ_i and ξ_i need not be distinct so that the following discussion includes the case of multiple poles and zeros. The real poles will be numbered according to their decreasing values; that is, $0 > \rho_1 \geq \rho_2 \geq \dots \geq \rho_m$. Thus, if a real pole of multiplicity r occurs at $s = a$ then r of the ρ_i will equal a . Moreover, all the zeros and the complex poles will be denoted by γ_i and numbered according to the decreasing values of their real parts; that is, denoting the real part of γ_i by α_i , the γ_i are numbered according to $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ where $n = h + g + q$. Zeros and complex poles having the same real part are numbered in any fashion. As in the case of real poles, if the multiplicity of a zero or complex pole is r , it is counted r times.

Furthermore, let $-G(\sigma)$ be the logarithmic derivative of $F(\sigma)$.

$$\begin{aligned}
 G(\sigma) &= -\frac{d}{d\sigma} \log F(\sigma) = -\frac{F'(\sigma)}{F(\sigma)} \\
 (4) \quad &= \sum_{i=1}^m \frac{1}{\sigma - \rho_i} + \sum_{i=1}^q \frac{1}{\sigma - \xi_i} - \sum_{i=1}^h \frac{1}{\sigma - \eta_i} - \sum_{i=1}^g \frac{1}{\sigma - \nu_i} \\
 &= \sum_{i=1}^m \frac{1}{\sigma - \rho_i} + \sum_{i=1}^n \frac{\pm 1}{\sigma - \gamma_i}.
 \end{aligned}$$

In the last expression, the plus sign is used if γ_i is a pole and a minus sign is used if γ_i is a zero.

The following lemma, which is due to Aaron and Segers [8], will be needed.

LEMMA 1. *Let all the poles and zeros of $F(s)$ have negative real parts. If $G(\sigma)$ is completely monotonic in $0 \leq \sigma < \infty$, then $F(\sigma)$ is also completely monotonic in the same interval.*

PROOF. Since all the poles and zeros of $F(s)$ have negative real parts and are either real or occur in complex conjugate pairs, it follows that $F(\sigma) > 0$ for $\sigma \geq 0$. Thus, from (4), $-F'(\sigma)$ is non-negative in $\sigma > 0$ if $G(\sigma)$ is non-negative there. Moreover, differentiating $-F'(\sigma)$ p times,

$$(5) \quad (-1)^{p+1} F^{(p+1)}(\sigma) = \sum_{k=0}^p \binom{p}{k} [(-1)^{p-k} F^{(p-k)}(\sigma)] [(-1)^k G^{(k)}(\sigma)].$$

Assuming that $(-1)^{p-k} F^{(p-k)}(\sigma)$ is non-negative for $\sigma \geq 0$ and for $k=0, 1, \dots, p$, it follows that it is also non-negative in this interval for $k=p-1$ since $G(\sigma)$ is completely monotonic by hypothesis. Therefore, the lemma is proved by induction.

The following fact will also be required.

LEMMA 2. *If $x \geq 0$, $y_i \geq 0$, and $x \geq \sum_{i=1}^n y_i$, then, for $k=1, 2, 3, \dots$,*

$$(x+1)^k + n - 1 \geq \sum_{i=1}^n (y_i + 1)^k.$$

The conclusion of this note may be stated as follows.

THEOREM. *Let $a(t)$ be a real, bounded function of the real variable t defined in the interval, $0 \leq t < \infty$, and let its Laplace-Stieltjes transform $F(s)$ be a rational function as given by (2) where $m \geq n = h + g + q$. Also, let the largest real pole ρ_1 satisfy $\alpha_i \leq \rho_1 < 0$ where α_i is the real part of any zero or complex pole and $i=1, 2, \dots, n$. Furthermore, let $A = \alpha_n$ if $\alpha_n \leq \rho_n$ and let $A = \rho_n$ if $\rho_n < \alpha_n$. If*

$$(6) \quad \sum_{i=1}^n \alpha_i \leq \rho_1 + (n-1)A,$$

than $a(t)$ is nondecreasing.

PROOF. By Lemma 1 and Bernstein's theorem, it is sufficient to show that $G(\sigma)$ is completely monotonic in $0 \leq \sigma < \infty$. Consider

$$(7) \quad (-1)^k G^{(k)}(\sigma) = k! \left\{ \sum_{i=1}^m \frac{1}{(\sigma - \rho_i)^{k+1}} + \sum_{i=1}^q \frac{1}{(\sigma - \xi_i)^{k+1}} - \sum_{i=1}^h \frac{1}{(\sigma - \eta_i)^{k+1}} - \sum_{i=1}^g \frac{1}{(\sigma - \nu_i)^{k+1}} \right\}.$$

That (7) is non-negative for $\sigma \geq 0$ and for $k=0, 1, 2, \dots$ may be shown as follows.

The inequality (6) may be rewritten as

$$\rho_1 - A \geq \sum_{i=1}^n (\alpha_i - A).$$

The quantities $\alpha_i - A$ are non-negative. Thus, for $\sigma \geq 0$,

$$\frac{\rho_1 - A}{\sigma - \rho_1} \geq \sum_{i=1}^n \frac{\alpha_i - A}{\sigma - \rho_1} \geq \sum_{i=1}^n \frac{\alpha_i - A}{\sigma - \alpha_i}.$$

Invoking Lemma 2, for $k=1, 2, 3, \dots$,

$$\left(\frac{\rho_1 - A}{\sigma - \rho_1} + 1 \right)^k + n - 1 \geq \sum_{i=1}^n \left(\frac{\alpha_i - A}{\sigma - \alpha_i} + 1 \right)^k.$$

Hence,

$$\frac{1}{(\sigma - \rho_1)^k} + \frac{n-1}{(\sigma - A)^k} \geq \sum_{i=1}^n \frac{1}{(\sigma - \alpha_i)^k}.$$

Now, $A \leq \rho_i$ for $i=2, 3, \dots, n$, so that the last inequality may be replaced by

$$\sum_{i=1}^n \frac{1}{(\sigma - \rho_i)^k} \geq \sum_{i=1}^n \frac{1}{(\sigma - \alpha_i)^k}.$$

For a pair of complex conjugate poles or zeros,

$$\frac{2}{(\sigma - \alpha_i)^k} > \left| \frac{1}{(\sigma - \gamma_i)^k} + \frac{1}{(\sigma - \bar{\gamma}_i)^k} \right|.$$

Using this inequality and the fact that $m \geq n$, it is readily seen that (7) is non-negative for $\sigma \geq 0$ and for $k=0, 1, 2, \dots$. Thus, $G(\sigma)$ is completely monotonic in the same interval and the theorem is established.

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ON PROJECTIONS OF SEPARABLE SUBSPACES OF (m) ONTO (c)

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1. There exists no bounded projection of the Banach space (m) of bounded real sequences onto the space (c) of convergent sequences [1] or onto the space (c_0) of null sequences [2]. It has been shown however by Sobczyk [2] that if B is a separable subspace of (m) properly containing (c_0), then there exists a projection P of norm 2 of B onto (c_0).

In the present paper we show that if B is a separable subspace of (m) containing (c), then there is a projection Q of B onto (c) with $\|Q\| \leq 3$. We then prove a theorem giving a lower bound for the norms of projections onto (c) of spaces of the form $(c) + (x)$, where (x) is the one-dimensional subspace determined by an element $x \notin (c)$. Using this result, we exhibit for each $n > 1$ a subspace B_n determined by (c) and $n - 1$ other elements of (m), such that the minimum of the norms of projections of B_n onto (c) is $3 - 2n^{-1}$. It follows that there exists a separable subspace $B \supset (c)$ such that any projection of B onto (c) has norm at least 3.

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