ON SURFACES WITH THE REPRESENTATION $z=f(x, y)$

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Introduction. A special case of the main result of this paper gives conditions under which a surface has a representation $z=f(x, y)$ in some Cartesian coordinate system with f a single-valued function or, more precisely, conditions which imply that there exists a line l with the property that any line l' parallel to l intersects the surface in at most one point.

Suppose for a moment that we are dealing with a surface in 3-dimensional Euclidean space which has a tangent plane everywhere. One of the conditions which we will use says that the set of normal vectors of the surface has a spherical image with a sufficiently small diameter. (By the spherical diameter of A, we mean, as usual, the supremum over all pairs of points $\boldsymbol{p}$ and $\boldsymbol{q}$ in A, of the spherical distance between $\boldsymbol{p}$ and $\boldsymbol{q}$, where this latter number is the length of the shorter great circular arc which has $\boldsymbol{p}$ and $\boldsymbol{q}$ for its endpoints.) The example of a helical ramp shows that it is not sufficient *merely* to require that the spherical image has a small diameter. In fact, we shall also require that the surface is the entire boundary of an open set. Of course, the result will be interesting only for open sets which are unbounded.

Pseudo-smooth surfaces. The restriction to surfaces possessing everywhere a tangent plane is unnecessarily strong. For example, we do not wish to exclude all polyhedral surfaces.

In the following G denotes an open set in Euclidean n-space and S denotes the boundary of G. Any straight line l intersects an open set G in a set which is open in the relative topology of l, i.e., $l\cap G$ is always a countable collection of disjoint open intervals. If l is a *directed* line we may speak of the right or left-hand end points of these intervals, which must, of course, belong to S, the boundary of G. If $\boldsymbol{p}$ is a right-hand end-point of $l\cap G$ then the directed line l is called an *exit line* with respect to $\boldsymbol{p}$, and a vector $\boldsymbol{v}$ of unit length which determines the direction of l is called an *exit vector* at $\boldsymbol{p}$.

If all of the exit vectors at $\boldsymbol{p}$ are contained in a closed half-space, i.e., if there exists a vector $\boldsymbol{n}$ of unit length such that for every exit vector $\boldsymbol{v}$ at $\boldsymbol{p}$ we have the inner product $(\boldsymbol{n}, \boldsymbol{v}) \geqq 0$, we say that $\boldsymbol{n}$ is

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a *pseudo-normal* of S at $\boldsymbol{p}$. This concept is a generalization of the notion of a tangent plane at $\boldsymbol{p}$. The pseudo-normal at a point need not be unique. Of course, at any point of S where no exit vectors exist, any unit vector $\boldsymbol{n}$ will serve as a pseudo-normal. If a pseudo-normal exists for every point of S, we say that S is *pseudo-smooth.* If a collection of vectors N contains at least one pseudo-normal for each point of S, we say N is a *system of pseudo-normals* for S.

This concept is illustrated by the following properties which are stated without proof:

PROPERTY 1. The boundary of any open convex set is pseudo-smooth. If $\boldsymbol{p}$ is a point on the boundary, then an outward unit normal to a supporting hyperplane through $\boldsymbol{p}$ will be a pseudo-normal at $\boldsymbol{p}$.

PROPERTY 2. Let G be the interior of a convex polyhedral set, and let S be the boundary of G. Then the set of outward unit normals to the faces of G constitutes a system of pseudo-normals for S.

PROPERTY 3. If S, the boundary of an open set G, has a tangent plane everywhere, then S is pseudo-smooth. An outward unit normal at a point gives the unique pseudo-normal at that point.

THEOREM 1. *Let S, the boundary of an open set G, be a pseudo-smooth subset of Euclidean n-space with a system of pseudo-normals N which is contained in an open hemisphere of the unit sphere S^{n-1}. Then there exists a line l such that any line l' parallel to l intersects S in at most one point.*

PROOF. Since N is contained in a hemisphere, there exists a vector $\boldsymbol{m}$ with the property that for any $\boldsymbol{n}$ in N we have $(\boldsymbol{n}, \boldsymbol{m})>0$. Then $-\boldsymbol{m}$ cannot be an exit vector for any point of S, since then we would have $(\boldsymbol{n}, -\boldsymbol{m}) = -(\boldsymbol{n}, \boldsymbol{m})>0$ for some $n \in N$.

Thus any line l with the direction of $\boldsymbol{m}$ intersects S at most once. For, from the above, $l \cap G$ consists of a system of disjoint intervals with no left-hand end points.

It remains only to give a convenient test for establishing that N is contained in an open hemisphere. The author is indebted to L. E. Dubins for conversations and correspondence concerning this theorem.

THEOREM 2. *Let N be a subset* [*connected subset*] *of the sphere S^{n-1} of points in Euclidean n-space of unit distance from the origin, and let d_n $(=\cos^{-1}(-1/n))$ be the spherical diameter of a regular n-simplex with vertices on S^{n-1}. Let the diameter of N be less than d_n* [*less than d_{n-1}*]. *Then the closure of N (and hence N itself) is contained in an open hemisphere of S^{n-1}.*

The proof of this theorem depends on the following known result (see [2]) which will be stated as a lemma.

LEMMA. *Any set of* $n+1$ *or fewer points of* S^{n-1} *which convexly covers the origin has diameter* $>d_n$.

PROOF OF THEOREM 2. Since diam $N<d_n$, we have diam $\overline{N}<d_n$, where $\overline{N}$ denotes the closure of N. By the lemma, $\overline{N}$ does not convexly cover $\mathbf{0}$, since any point convexly covered by $\overline{N}$ is a convex linear combination of $n+1$ or fewer points of N. Hence, $\overline{N}$ may be separated from $\mathbf{0}$ by plane. Thus $\overline{N}$ is contained in an open hemisphere of S^{n-1}.

If N is connected, then a point convexly covered by N is convexly covered by n or fewer points of N (see [1, p. 9]). If n points of N convexly cover $\mathbf{0}$, then these n points belong to an $n-2$-sphere centered at the origin, and the above argument shows that diam N $<d_{n-1}$ implies that $\overline{N}$ is contained in an open hemisphere.

We obtain the following generalization of Theorem 2 by observing simply that if N is contained in a half-space which does not contain the origin, then any affine image of N has the same property.

THEOREM 2′. *Let* P *be any real symmetric positive definite matrix Let* N *be any subset [connected subset] of vectors* $\mathbf{x}$ *in Euclidean n-space such that* $(\mathbf{x}, P\mathbf{x})=1$. *Let the greatest lower bound of* $(\mathbf{x}, P\mathbf{y})$ *over all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* N *be greater than* $-1/n$ *[greater than* $-1/(n-1)$*]. Then* N *is contained in a closed half-space which does not contain the origin.*

PROOF. There exists a nonsingular real matrix S such that $P=S^TS$. Consider the set N' of vectors $\boldsymbol{u}$ such that $\boldsymbol{u}=S\mathbf{x}$ for some $\mathbf{x}$ in N. Then $(\boldsymbol{u}, \boldsymbol{u})=1$ for all $\boldsymbol{u}$ in N', and the greatest lower bound of $(\boldsymbol{u}, \boldsymbol{v})$ over all $\boldsymbol{u}$ and $\boldsymbol{v}$ in N' is greater than $-1/n$ [greater than $-1/(n-1)$]. By Theorem 2, N' is contained in a closed half space which does not contain the origin, and hence the same is true for N, which is an affine image of N'.

REFERENCES

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