ON THE SPACE-OF-LOOPS ISOMORPHISM

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For X an arc-wise connected topological space with base point x_0 , let ΩX be the space of loops on X ; that is, ΩX is the subset of the function space X^I consisting of all paths $\lambda: I \rightarrow X$ such that $\lambda(0) = \lambda(1) = x_0$, where I is the unit interval. Let $|\Omega|X$ be the geometric realization of the singular complex of ΩX ; $|\Omega|X$ is a CW-complex [5].

For arc-wise connected topological spaces X and Y with base points x_0 and y_0 respectively, denote by $\pi(X, Y)$ the set of homotopy classes of maps with base point of X into Y . $\pi(X, Y)$ is in fact an invariant of the homotopy types of X and Y , so we will use the most convenient representatives of the homotopy types in discussing $\pi(X, Y)$.

Recall that $\pi(X, Y)$ is known to have a natural group structure [3, p. 371] if it belongs to one of the following types:

- (1) X is a suspension space, or
- (2) Y is an H -space, or
- (3) the dimension of X is $\leq 2n-2$ where Y is $(n-1)$ -connected and either X is a CW-complex or X is compact and Y is a CW-complex.

Our first theorem adds an approximate dual to (3), just as (1) and (2) are dual:

THEOREM 1. *If X is an $(n-1)$ -connected CW-complex and $\pi_i(Y) = 0$ for $i > 2n-2$, then $\pi(X, Y)$ forms an abelian group, natural with respect to maps $X \rightarrow X'$ and $Y \rightarrow Y'$ where X' and Y' satisfy the same conditions.*

With reference to [3] and the above list, we will say the pair X, Y or the group $\pi(X, Y)$ is of type (4) under these conditions.

Peterson [6] has investigated the "dual" group of type (3) by means of the generalized suspension isomorphism. Continuing the duality, we show

THEOREM 2. *If X is an $(n-1)$ -connected CW-complex and $\pi_i(Y) = 0$ for $i > 2n-2$, then $\pi(X, Y)$ is naturally isomorphic to $\pi(|\Omega|X, \Omega Y)$.*

Theorem 2 can be interpreted as giving sufficient conditions for homotopy classes of maps of $|\Omega|X$ into ΩY to coincide with loop homotopy classes in the sense of Kan [4, p. 41].

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Theorem 1 is proved in §2, the group structure is investigated in relation to the other types and a natural homomorphism $\Omega_k: \pi(X, Y) \rightarrow \pi(\Omega X, \Omega Y)$ is defined. §3 will prove the "Space-of-loops isomorphism."

I would like to thank J. F. Adams for many helpful discussions concerning H -spaces.

2. Before proving Theorem 1, we establish:

LEMMA 1. *If Z is a CW-complex such that $\pi_i(Z) = 0$ for $i < n$ and $i > 2n - 2$ then Z is an H -space; in fact, there exists a space Z' such that Z has the homotopy type of $|\Omega| Z'$.*

LEMMA 2. *For such a Z , the H -space structure is unique up to homotopy.*

Evidently both of these are well-known, though I am unable to assign an author to either. Lemma 2 is proved in [9] and Lemma 1 can be deduced from arguments there. We give a brief proof of the latter to lend insight into our basic approach.

Let Z be represented by a finite Postnikov system $\{Z_i\}$, $i = n, \dots, 2n - 2$. That is, $\pi_j(Z_i) = \pi_j(Z)$ for $j \leq i$, and each Z_i is the geometric realization of the singular complex of a fibre space over Z_{i-1} with fibre $K(\pi_i(Z), i)$. The system and Z are determined by $\pi_i(Z)$ and k -invariants $k^{i+1} \in H^{i+1}(Z_{i-1}; \pi_i(Z))$. We prove Lemma 1 by induction on j where $\pi_i(Z) = 0$, $i > j$.

Since $Z_n = K(\pi_n(Z), n)$, Lemma 1 is true for $j = n$, for $K(\pi_n(Z), n)$ has the same homotopy type as $|\Omega| K(\pi_n(Z), n+1)$.

Induction, $j-1 \Rightarrow j$ for $j \leq 2n-2$: By induction, $Z_{j-1} \simeq |\Omega| Z'_{j-1}$ for some Z'_{j-1} . Thus $k^{i+1} \in H^{i+1}(|\Omega| Z'_{j-1}; \pi_j(Z))$. Now Z'_{j-1} is n -connected, so we have the suspension isomorphism in cohomology:

$$\sigma: H^{i+2}(Z'_{j-1}; \pi_j(Z)) \approx H^{i+1}(|\Omega| Z'_{j-1}; \pi_j(Z))$$

for $0 < j+1 < 2n$ [10, p. 94; 8]. Thus for $j \leq 2n-2$, $k^{i+1} = \sigma' k^{i+2}$ for some $'k^{i+2} \in H^{i+2}(Z'_{j-1}; \pi_j(Z))$, and so $Z_j \simeq |\Omega| Z'_j$ where Z'_j is the fibre space determined by Z'_{j-1} and $'k^{i+2}$.

PROOF OF THEOREM 1. Using the Postnikov system, we can represent Y as a fibre space with fibre Z and base W where $\pi_i(Z) = \pi_i(Y)$, $i \geq n$ and $= 0$, $i < n$, and $\pi_i(W) = \pi_i(Y)$, $i < n$ and $= 0$, $i \geq n$. Let $|Z|$ be the geometric realization of the singular complex of Z . By Lemma 1, $|Z|$ is an H -space so that $\pi(X, |Z|)$ is a group, natural for maps $X' \rightarrow X$ and H -maps $|Z| \rightarrow Z'$.

Consider the map $\phi: \pi(X, |Z|) \rightarrow \pi(X, Y)$ induced by the inclusion $i: Z \rightarrow Y$ of Z as the fibre through y_0 .

ϕ is onto: Since X is $(n-1)$ -connected, we can assume $X^{n-1} = x_0$. If $f: X \rightarrow Y$ then $f(X^{n-1}) \subset i(Z)$. Since $\pi_n(Y, Z) \approx \pi_n(W) = 0$ we can deform X^n into $i(Z)$ and similarly for $i > n$.

ϕ is 1-1: That is if $f: X \rightarrow Z$ is homotopic in Y to $g: X \rightarrow Z$ then f is homotopic to g in Z . Since all the homotopies are relative to x_0 , $f|X^{n-1} \simeq g|X^{n-1}$ in Z . Corresponding to each k -cell of X , there is a $k+1$ -cell of $X \times I$. Again $\pi_{k+1}(Y, Z) \approx \pi_{k+1}(W) = 0$ for $k \geq n-1$ so we can deform the homotopy, skeleton by skeleton, into $i(Z)$. (We have actually shown ϕ is 1-1 for X $(n-2)$ -connected.) Although Z may look like a space of loops on different spaces, the multiplications coincide by Lemma 2 so the group $\pi(X, Z)$ is uniquely determined. Since Y does not determine Z uniquely, we need to show further that the induced group structure on $\pi(X, Y)$ is independent of the choice of Z . This will be done while studying the naturality.

Naturality: That is, maps $f: Y \rightarrow Y'$ where $\pi_i(Y') = 0$ for $i > 2n-2$ or $g: X' \rightarrow X$ where X' is an $(n-1)$ -connected CW-complex induce homomorphisms $f_*: \pi(X, Y) \rightarrow \pi(X, Y')$ and $g^*: \pi(X, Y) \rightarrow \pi(X', Y)$. Consider first g^* . The diagram

$$\begin{array}{ccc} \pi(X, Y) & \xrightarrow{g^*} & \pi(X', Y) \\ \uparrow \phi & & \uparrow \phi \\ \pi(X, Z) & \xrightarrow{g^*} & \pi(X', Z) \end{array}$$

is commutative, since ϕ is induced by i and $(i \circ h) \circ g = i \circ (h \circ g)$ for $h: X \rightarrow Z$. ϕ is a homomorphism by definition and $g^*: \pi(X, Z) \rightarrow \pi(X', Z)$ is known to be a homomorphism, hence $g^*: \pi(X, Y) \rightarrow \pi(X', Y)$ is a homomorphism.

Before considering f_* , we give an alternative description of the group structure on $\pi(X, Y)$. J. H. C. Whitehead has proved [1, p. 288] that an $(n-1)$ -connected CW-complex of dimension $\leq 2n-1$ is of the same homotopy type as a suspension space. Thus $\pi(X^{2n-1}, Y)$ is a group of type (1). Consider $\eta: \pi(X, Y) \rightarrow \pi(X^{2n-1}, Y)$ induced by the inclusion $X^{2n-1} \subset X$. Just as for ϕ , we prove

η is onto: $f: X^{2n-1} \rightarrow Y$ can be extended to all of X since $\pi_i(Y) = 0$ for $i > 2n-2$.

η is 1-1: The obstructions to extending a homotopy vanish for the same reason. (In fact, η is 1-1, if $\pi_i(Y) = 0$ for $i > 2n-1$.) Thus $\pi(X, Y)$ has a group structure induced by η from that on $\pi(X^{2n-1}, Y)$.

Since X^{2n-1} is not uniquely determined by the homotopy type of X and further may be the suspension of two different spaces, the group $\pi(X, Y)$ has not yet been shown to be uniquely determined. This will

follow from our next lemma which compares, for any given choices of X^{2n-1} as a suspension of a particular space and of Z , the corresponding structures on $\pi(X, Y)$:

LEMMA 3. η and ϕ induce naturally isomorphic group structures on $\pi(X, Y)$.

Consider the diagram

$$\begin{array}{ccc} \pi(X, Y) & \xrightarrow{\eta} & \pi(X^{2n-1}, Y) \\ \uparrow \phi & & \uparrow \phi' \\ \pi(X, |Z|) & \xrightarrow[\eta']{} & \pi(X^{2n-1}, |Z|) \end{array}$$

which is clearly commutative. η', ϕ' are isomorphisms of both group structures, by naturality or by definition. Since the two structures agree in $\pi(X^{2n-1}, |Z|)$ [3], the induced ones agree in $\pi(X, Y)$. Since this is true for all choices of X^{2n-1} and Z , $\pi(X, Y)$ is well-defined and is an invariant of the homotopy types of X and Y .

REMARK. Since Hilton has shown $\pi(S, H)$ is an abelian group for S a suspension space and H an H -space, $\pi(X, Y)$ of type (4) is abelian. This agrees with the previous empirical evidence that two distinct natural group structures on $\pi(X, Y)$ will coincide and be abelian.

Returning to $f_*: \pi(X, Y) \rightarrow \pi(X, Y')$, consider the commutative diagram

$$\begin{array}{ccc} \pi(X, Y) & \xrightarrow{f_*} & \pi(X, Y') \\ \eta \downarrow & & \downarrow \eta \\ \pi(X^{2n-1}, Y) & \xrightarrow[f_*]{} & \pi(X^{2n-1}, Y') \end{array}$$

corresponding to the situation for g^* and ϕ . η is an isomorphism on both sides and $f_*: \pi(X^{2n-1}, Y) \rightarrow \pi(X^{2n-1}, Y')$ is known to be a homomorphism, hence $f_*: \pi(X, Y) \rightarrow \pi(X, Y')$ is. This concludes the proof of Theorem 1.

Since we know that two definitions of group structure for $\pi(X, Y)$ coincide, we might investigate the possibilities if X, Y were also of type (1), (2) or (3). [Hilton has shown [3] that if X, Y belongs to two of types (1), (2) or (3), then the group structures coincide.]

LEMMA 4. If X, Y is of type (4) and also (1), (2) or (3) the group structures coincide.

Type (1). X is a suspension space. We note that the definitions of type (1) and (2) coincide if they are both present, so that they coincide in $\pi(X, |Z|)$. ϕ is a homomorphism of the suspension structure by naturality and of type (4) structure by definition, hence the definitions coincide in $\pi(X, Y)$.

Type (2). Y is an H -space. Dually to the above, we observe that the suspension and H -structures agree in $\pi(X^{2n-1}, Y)$ and hence the type (4) and H -structures agree in $\pi(X, Y)$. Alternatively, it is possible to show that $|i|: |Z| \rightarrow Y$ is an H -map.

Type (3). We have $\dim X \leq 2m-2$ and $Y(m-1)$ -connected. The situation is very simple for if $m \geq n$, $|Y|$ is an H -space and we reduce to type (2), whereas if $m \leq n$ X is a suspension space and we reduce to type (1).

In order to establish Theorem 2, we define $|\Omega|_{\#}: \pi(X, Y) \rightarrow \pi(|\Omega|X, \Omega Y)$. For $f: X \rightarrow Y$, and $\lambda \in \Omega X$, let $\Omega_{\#}(f)(\lambda) = f \circ \lambda$ considering λ as a map $I \rightarrow X$. There is a map $j: |\Omega|X \rightarrow \Omega X$ which induces isomorphisms of singular homology groups and homotopy groups. Define $|\Omega|_{\#}(f)$ as $\Omega_{\#}(f) \circ j$.

LEMMA 5. $|\Omega|_{\#}$ is a homomorphism if X, Y is of type (4).

$|\Omega|_{\#}$ is a homomorphism if $\Omega_{\#}$ and j^* are. The latter is a homomorphism by naturality of the H -structure, ΩY being an H -space. That $\Omega_{\#}$ is a homomorphism can be seen from the following diagram:

$$\begin{array}{ccc} \pi(X, Y) & \xrightarrow{\Omega_{\#}} & \pi(\Omega X, \Omega Y) \\ \phi \uparrow & & \uparrow \Omega\phi \\ \pi(X, Z) & \xrightarrow{\Omega_{\#}} & \pi(|\Omega|X, \Omega|Z|) \end{array}$$

$\Omega_{\#}'$ is a homomorphism since the loop multiplication on $|\Omega|Z|$ is homotopic to that induced from the H -structure on $|Z|$. $\Omega\phi$ is clearly a homomorphism since it is induced by $\Omega_{\#}(|i|): |\Omega|Z| \rightarrow \Omega Y$ which is an H -map. $\Omega\phi$ is in fact an isomorphism since $\pi_i(\Omega Y, \Omega|Z|) = 0$ for $i \geq n-1$ and $|\Omega|X$ is $(n-2)$ -connected. Thus $\Omega_{\#}$ is a homomorphism.

3. In terms of the previous paragraph, to prove Theorem 2, we need only show that $\Omega_{\#}': \pi(X, Z) \rightarrow \pi(|\Omega|X, \Omega|Z|)$ is an isomorphism. We will prove the theorem by induction on j where $\pi_i(Y) = 0$ for $i > j \leq 2n-2$.

Special case. $n=1$ so $\pi_i(Y) = 0$ for $i \geq 0$ which means $\pi(X, Y) = 0$ and $\pi(|\Omega|X, \Omega Y) = 0$.

For $n > 1$, we start with $j=n$ since Z is $(n-1)$ -connected. $|Z|$ is,

up to homotopy type, $K(\pi_n(Y), n)$ so our theorem reduces to the known result

$$H^n(X; \pi_n(Y)) \approx H^{n-1}(\Omega X; \pi_n(Y)).$$

Induction, $j-1 \Rightarrow j$ for $j \leq 2n-2$: For any fibre space E with fibre F over base B we have the following exact sequence [7, p. 199],

$$(1) \quad \begin{array}{ccccccc} \cdots & \rightarrow & \pi(X, \Omega^{n+1}B) & \rightarrow & \pi(X, \Omega^n F) & \rightarrow & \pi(X, \Omega^n E) \rightarrow \pi(X, \Omega^n B) \rightarrow \\ \cdots & \rightarrow & \pi(X, \Omega B) & \rightarrow & \pi(X, F) & \rightarrow & \pi(X, E) \rightarrow \pi(X, B). \end{array}$$

Applied to Z_j , of the Postnikov system of $|Z|$, we have

$$(2) \quad \cdots \rightarrow \pi(X, \Omega Z_{j-1}) \rightarrow \pi(X, K) \rightarrow \pi(X, Z_j) \rightarrow \pi(X, Z_{j-1})$$

where $K = K(\pi_j(Z), j)$, but since $|Z| \simeq |\Omega|Z'|$ we can extend this and, writing $W = |\Omega|X$, $R = Z_j$, $S = Z_{j-1}$ and $K' = K(\pi_j(Z), j+1)$, obtain

$$(3) \quad \begin{array}{ccccccccc} \cdots & \rightarrow & \pi(X, \Omega S) & \rightarrow & \pi(X, K) & \rightarrow & \pi(X, R) & \rightarrow & \pi(X, S) \rightarrow \pi(X, K') \rightarrow \cdots \\ & & a] \downarrow & & b] \downarrow & & c] \downarrow & & d] \downarrow & & e] \downarrow \\ \cdots & \rightarrow & \pi(W, \Omega^2 S) & \rightarrow & \pi(W, \Omega K) & \rightarrow & \pi(W, \Omega R) & \rightarrow & \pi(W, \Omega S) \rightarrow \pi(W, K) \rightarrow \cdots \end{array}$$

where $K'_j = K(\pi_j(Z), j+1)$.

All the sets involved are groups with horizontal arrows being homomorphisms. Lemma 5 applies to all the vertical arrows; they are homomorphisms $|\Omega|_#$. The canonical way in which all the homomorphisms are induced from maps makes all the squares commutative.

By induction we can assume that $a]$ is an isomorphism for $j \leq 2n-1$ (since $\pi_i(\Omega Z_{j-1}) = 0$, $i > j-2$) and that $d]$ is an isomorphism for $j < 2n-1$. A special case of our theorem is the following

SUSPENSION THEOREM. *If W is an $(n-1)$ -connected space, the "suspension" map*

$$\sigma: H^i(W; \pi) \rightarrow H^{i-1}(\Omega W; \pi)$$

is an isomorphism for $0 < i < 2n-1$ and a monomorphism for $i = 2n-1$. [10, p. 94; 8].

Thus we also have that $b]$ is an isomorphism for $0 < j < 2n-1$, monomorphism for $j = 2n-1$ and $e]$ is an isomorphism for $0 < j+1 < 2n-1$, monomorphism for $j+1 = 2n-1$. By the Five Lemma, $c]$ is an isomorphism for $j \leq 2n-2$.

REMARK. The final statement of the Suspension Theorem that σ is monomorphic for $i = 2n-1$ can also be generalized.

THEOREM 2a. *If X is an $(n-1)$ -connected CW complex, $\pi_i(Y) = 0$ for $i > 2n-1$ and some choice of $|Z|$ is an H -space, then $|\Omega|_#: \pi(X, Y) \rightarrow \pi(|\Omega|X, \Omega Y)$ is a monomorphism.*

For then we still have the exact sequences (7) (since $|Z|$ is an H -space, the map $Z_{j-1} \rightarrow K'_j$ is an H -map and so $\pi(X; Z_{j-1}) \rightarrow \pi(X, K'_j)$ is a homomorphism) and the half of the Five Lemma [2, Lemma 4.4] necessary to prove $c]$ monomorphic for $j = 2n - 1$ requires only that $a]$ be onto and that $b]$ and $d]$ be monomorphic. Modifying Theorem 2 to Theorem 2a throughout, the induction goes through.

The extra assumption that $|Z|$ is an H -space is necessary to make $\pi(X, Y)$ a group and the top line of (3) an exact sequence of homomorphisms.

If it is desired to have Theorem 2 in a more symmetric form, we need only observe that $\pi(|\Omega|X, \Omega Y)$ is isomorphic with $\pi(|\Omega|X, |\Omega|Y)$ since $j: |\Omega|Y \rightarrow \Omega Y$ induces isomorphisms of the homotopy groups and $|\Omega|X$ is a CW-complex. Whether $\pi(|\Omega|X, \Omega Y)$ is isomorphic with $\pi(\Omega X, \Omega Y)$ is related to the general problem of giving ΩX as a CW-complex, up to homotopy type.¹

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¹ *Added in proof.* This problem has recently been solved by J. Milnor.