A FREE BOUNDARY PROBLEM FOR PSEUDO-ANALYTIC FUNCTIONS[1]

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1. **Introduction.** In this note we consider free boundary problems for equations of the form

$$\Delta\phi = g(x, y, \phi_x, \phi_y) \tag{1}$$

and

$$W_{\bar{z}} = \rho(z, W). \tag{2}$$

In the latter equation W and ρ are complex, and

$$W_{\bar{z}} = \frac{1}{2}(W_x + iW_y).$$

We shall show that a simple method can reduce free boundary problems for these equations to the same problem for harmonic or analytic functions. The corresponding problems for analytic functions can readily be handled by conformal mapping techniques.

We employ a method developed by Bers [1] in the study of pseudo-analytic functions. In order to illustrate the ideas, we have picked an example which has no technical difficulties. More involved applications will be considered in subsequent publications.

2. **The problem for equation** (1). Let Γ be a simple closed curve with continuously turning tangent and assume that its interior D contains the origin. Assume that there is a (possibly multiple valued) function ϕ having continuous second derivations in $D-[0]$ and such that

1. ϕ satisfies equation (1) in $D-[0]$,
2. ϕ has continuous first derivatives in $\overline{D}-[0]$ and

$$\phi_x^2 + \phi_y^2 = 1 \text{ on } \Gamma,$$

3. The normal derivative of ϕ vanishes on Γ,
4. $(\phi_x^2+\phi_y^2)(x^2+y^2)\to\lambda^2\neq 0$ as $(x, y)\to 0$.

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If Γ and ϕ satisfy conditions 1–4, we shall say that they solve problem I.

Our result concerning problem I is the following

THEOREM 1. *Assume that $g(x, y, u, v)$ is Hölder continuous in all arguments in every closed subdomain of $0<x^2+y^2<\infty$, $0<u^2+v^2<\infty$ and that for $x^2+y^2\leqq R^2$*

$$|g(x, y, u, v)| \leqq N_R(|u| + |v|),$$

where the constant N_R depends only on R. Then there exists a positive constant λ_0 depending only on N_R such that problem I *has a solution for all $0<\lambda<\lambda_0$. If $g\equiv 0$, then $\lambda_0=\infty$, Γ is the circle $x^2+y^2=\lambda^2$, and $\phi=\pm\tan^{-1}(y/x)$.*

Theorem 1 follows from a more general theorem given in the next section.

3. **The problem for equation** (2). Now assume that Γ and ϕ solve problem I and set

$$W = \phi_x - i\phi_y = 2\phi_z.$$

Then W has continuous derivations in $D-[0]$ and has the following properties

1°. W satisfies equation (2) with

$$\rho(z, W) = g(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} W, -\operatorname{Im} W), \tag{3}$$

2°. W is continuous in $\overline{D}-[0]$ and

$$|W| = 1 \text{ on } \Gamma,$$

3°. If $n=n_x+in_y$ is normal to Γ,

$$\operatorname{Re} Wn = 0 \text{ on } \Gamma,$$

4°. $|Wz|\to\lambda>0$ as $z\to 0$.

If Γ and W satisfy conditions 1°–4° with any complex function $\rho(z, W)$ not necessarily satisfying (3), we shall say that Γ and W solve problem II.

Concerning problem II we shall prove the following result

THEOREM 2. *Let $\rho(z, W)$ be Hölder continuous in z and W in every closed subdomain of $0<|z|<\infty$, $0<|W|<\infty$, and assume that for $|z|\leqq R$*

$$|\rho(z, W)| \leqq N_R|W|,$$

where the constant N_R depends only on R. Then there is a positive con-

stant λ_0 *depending only on* N_R *such that problem* II *has a solution for all* $0<\lambda<\lambda_0$. *If* $\rho\equiv 0$, *then* $\lambda_0=\infty$, Γ *is the circle* $|z|=\lambda$, *and* $W=\pm i\lambda z^{-1}$.

The proof of Theorem 2 will be given in the next section.

Now assume that Γ and W solve problem II with $\rho(z, W)$ a real function. We shall show how to construct a solution of problem I. In fact if we set

$$W = u - iv,$$

the condition

$$\operatorname{Im} W_{\bar{z}} = 0$$

implies

$$u_y = v_x.$$

This means that the possibly multiple valued function

$$\phi(x, y) = \frac{1}{2}\int_{Z_1}^{z} udx + vdy = \frac{1}{2} \operatorname{Re} \int_{Z_1}^{z} Wdz$$

exists in $D-[0]$, where $Z_1\neq 0$ is some fixed point in D. Moreover ϕ has continuous second derivatives in $D-[0]$ and

$$W = 2\phi_z.$$

Hence

$$\Delta\phi = 4\phi_{z\bar{z}} = 2\rho(x+iy, \phi_x - i\phi_y) \equiv g(x, y, \phi_x, \phi_y). \tag{4}$$

Since the normal derivative $\partial\phi/\partial n$ equals

$$\frac{\partial\phi}{\partial n} = un_x + vn_y = \operatorname{Re} Wn,$$

Γ and ϕ solve problem I. Since the hypotheses assumed for $\rho(z, W)$ in Theorem 2 imply those assumed for $g(x, y, u, v)$ in Theorem 1 when g is defined as in (4), Theorem 1 follows immediately from Theorem 2.

4. **The proof of Theorem** 2. That

$$\Gamma\colon\ |z| = \lambda, \qquad w = \pm i\lambda z^{-1}$$

is a solution of problem II with $\rho\equiv 0$ is easily checked. It also is the only solution with smooth boundary. For let Γ and w be any such solution, and let $f(z)$ be the analytic function which maps the interior D of Γ onto $|z|<\lambda$ in such a way that $f(0)=0$. If $n=n_x+in_y$ is normal to Γ,

$$n = \frac{z}{\lambda} f'(z) = e^{i\theta} f'(\lambda e^{i\theta})$$

on Γ and hence

$$\operatorname{Re} zw(f(z))f'(z) = 0$$

on $|z| = \lambda$. Since

$$|zw(f(z))f'(z)| \sim |fw(f)| \to \lambda \tag{5}$$

as $z \to 0$, the function $zw(f(z))f'(z)$ is analytic in $|z| < \lambda$ and imaginary on the boundary. Thus it equals an imaginary constant, and this constant is $\pm i\lambda$ by (5). Since $|w(f(z))| = 1$ on $|z| = \lambda$, $|f'(z)| = 1$ there. Since $f'(z) \neq 0$ in $|z| < \lambda$, we must have $f'(z) = e^{ic}$. Hence $fw(f) = \pm i\lambda$ and Γ is the circle $|f| = \lambda$.

Now consider complex functions $s(z)$ defined in $|z| \leqq \lambda$. If $s(z)$ is bounded there, set

$$\|s\| = \max_{|z| \leqq \lambda} |s(z)|.$$

If $s(z)$ is Hölder continuous in $|z| \leqq \lambda$ with exponent α, set

$$|s|_\alpha = \operatorname*{lub}_{|z_1| \leqq \lambda;\, |z_2| \leqq \lambda} \frac{|s(z_1) - s(z_2)|}{|z_1 - z_2|^\alpha},$$

$$\|s\|_\alpha = \|s\| + |s|_\alpha.$$

We denote the space of all $s(z)$ such that

$$\|s\|_\alpha < \infty$$

by C_α. Next we define

$$As(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\lambda e^{i\theta} + z}{\lambda e^{i\theta} - z} \operatorname{Re} s(\lambda e^{i\theta}) d\theta + i \operatorname{Im} s(0).$$

It is easily recognized that As is the analytic function in $|z| < \lambda$ such that

$$\operatorname{Re}(s - As) = 0 \quad \text{on } |z| = \lambda,$$
$$\operatorname{Im}(s - As) = 0 \quad \text{at } z = 0.$$

By a theorem of Privaloff [2] and Bers [1] we have

LEMMA 1. *If $s \in C_\alpha$, then $As \in C_\alpha$ and*

$$|As|_\alpha \leqq K_\alpha |s|_\alpha$$

where the constant K_α depends only on α.

Since A is linear, Lemma 1 shows that A is a continuous transformation of C_α into itself.

Next, consider the operation

$$T\sigma(z) = -\frac{1}{\pi}\int\int_{|\zeta|<\lambda}\frac{\sigma(\zeta)d\xi d\eta}{\zeta - z}$$

where $\zeta = \xi + i\eta$. Concerning T we have

LEMMA 2. *If $\sigma(z)$ is bounded in $|z| \leqq \lambda$, then $T\sigma \in C_\beta$ for all $\beta < 1$ and*

$$\|T\sigma\| \leqq 2\lambda\|\sigma\|,$$
$$|T\sigma|_\beta \leqq M_\beta\|\sigma\|$$

where the constant M_β depends only on β.

Lemma 2 shows that T is completely continuous in any C_β, $\beta < 1$. Since it is linear, it is also continuous.

LEMMA 3. *If $\sigma(z)$ is Hölder continuous in every bounded subdomain of $0 < |z| < \lambda$, $T\sigma$ has continuous derivatives in $0 < |z| < \lambda$ and*

$$(T\sigma)_{\bar{z}} = \sigma$$

there.

The proofs of Lemmas 2 and 3 may be found in [1].

We are now ready for the

PROOF OF THEOREM 2. Let B_M be the set of all $s \in C_\alpha$ such that

$$\|s\|_\alpha \leqq M.$$

If $s \in B_M$, let $f(z)$ be the analytic function defined by

$$f(0) = 0,$$
$$f'(z) = e^{As + iAis}.$$

Now by Lemma 1

$$|As - s|_\alpha \leqq (K_\alpha + 1)M$$

and since $As - s$ is imaginary on the boundary and real at $z = 0$,

$$\|As - s\| \leqq 2M\lambda^\alpha(K_\alpha + 1).$$

Similarly

$$\|iAis + s\| \leqq 2M\lambda^\alpha(K_\alpha + 1).$$

Hence

$$\|As + iAis\| \leqq 4M\lambda^\alpha(K_\alpha + 1) \equiv L$$

and

$$|f'(z)| \leqq e^L$$

in $|z| \leqq \lambda$. Since $f(0)=0$

$$|f(z)| \leqq \lambda e^L \equiv \nu. \tag{6}$$

Now by hypothesis

$$\rho(z, w) = \tau(z, w)w \tag{7}$$

where

$$|\tau(z, w)| \leqq N_R \tag{8}$$

in $|z| \leqq R$ and $\tau(z, w)$ is Hölder continuous in every closed subdomain of $0<|z|<\infty$, $0<|w|<\infty$. Set

$$\begin{aligned} S(z) &= T\{[f'(z)]^{-}\tau(f(z), w_0(z)e^{s-As})\} \\ &= T_0 s(z), \end{aligned}$$

where $[f'(z)]^{-}$ denotes the complex conjugate of $f'(z)$ and $w_0 = i\lambda z^{-1}$. Applying Lemma 2 to (6) and (8), we see that

$$\begin{aligned} ||S|| &\leqq 2\nu N_\nu, \\ |S|_\beta &\leqq M_\beta e^L N_\nu \end{aligned}$$

for $\beta<1$. If we take $\alpha<\beta<1$, we have

$$||S||_\alpha \leqq (2\lambda + \lambda^{\beta-\alpha}M_\beta)e^L N_\nu.$$

We now pick λ so small that

$$||S||_\alpha \leqq M,$$

i.e., $S\in B_M$. Thus T_0 maps B_M into itself. By Lemmas 1 and 2 it is continuous and completely continuous in C_α. Hence by the Schauder fixed point theorem [3] there is a function $s\in B_M$ such that

$$s(z) = T_0 s(z).$$

Lemma 3 shows that $s(z)$ has continuous derivatives in $|z|<\lambda$ and that

$$s_{\bar z} = [f'(z)]^{-}\tau(f(z), w_0 e^{s-As}). \tag{9}$$

Let

$$z = h(\zeta)$$

be the inverse mapping of $\zeta = f(z)$ and set

$$w(\zeta) = w_0(h(\zeta))e^{t-At} \tag{10}$$

where $t(\zeta)=s(h(\zeta))$. If Γ is the image of $|z|=\lambda$ under the map $f(z)$, we claim that Γ and w solve problem II.

In fact, by (10)

$$w_{\bar{z}} = wt_{\bar{\zeta}}$$

and from (9)

$$t_{\bar{\zeta}} = s_{\bar{z}}[h'(z)]^{-} = \tau(\zeta, w)$$

and hence by (7)

$$w_{\bar{z}} = \rho(\zeta, w).$$

In addition $s-As$ is imaginary on $|z|=\lambda$. Hence

$$|w| = |w_0| \cdot |e^{t-At}| = 1$$

on Γ. Moreover, if $n=n_x+in_y$ is normal to Γ

$$n = \frac{z}{\lambda} f'(z)$$

and hence

$$\begin{aligned} wn &= w_0 e^{t-At} \cdot \frac{z}{\lambda} \cdot e^{At+iAit} \\ &= ie^{t+iAit}. \end{aligned}$$

But it is easily checked that $s+iAis$ is real on $|z|=\lambda$ and hence

$$\operatorname{Re} wn = 0$$

on Γ. Finally,

$$\begin{aligned} |w\zeta| &\sim |wf'(z)z| \sim |w_0 e^{s-As} \cdot C^{As+iAis} z| \\ &\sim |\lambda e^{s+iAis}| = \lambda \end{aligned}$$

as $\zeta \to 0$, since $s+iAis$ is imaginary at $z=0$. Hence Γ and w solve problem II. This completes the proof.

Bibliography

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