ON APPROXIMATION BY NONVANISHING FUNCTIONS

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1. **The measure of approximation.** Let the function $f(z)$ be analytic and uniformly limited and have k zeros interior to a simply connected region G of the plane of the complex variable $z=x+iy$. The set of functions analytic interior to G and which vanish either identically or not at all interior to G form a closed set S [1, p. 343]. For every function $F(z)$ uniformly limited and of class S in G we set

$$M(F) = \text{l.u.b.}[\,|F(z) - f(z)|\,,\ z \text{ in } G]$$

and let M denote the greatest lower bound of all $M(F)$. Walsh has shown [1, pp. 344–346] that there exists a function in S, call it $F^*(z)$, for which $M(F^*)=M$. In certain situations he has found the precise value of M as well as functions $F^*(z)$ of best approximation. In some instances he has exhibited an infinity of functions $F^*(z)$ of best approximation.

The purpose of the present paper is (1) to put an appraisal on M, (2) to present two theorems on the number of zeros in G of functions which approximate closer to $f(z)$ than the lower appraisal on M.

2. **Appraisal of** M. Consider a region G, a function $f(z)$, the set S of functions $F(z)$ and the measure M of best approximation to $f(z)$ by functions of class S, all as described above in §1. Let the k zeros of $f(z)$ in G occur as follows: k_1 zeros at $z=a_1$, k_2 zeros at $z=a_2, \cdots, k_m$ zeros at $z=a_m$, where $k_1+k_2+\cdots+k_m=k$. Let R denote the Riemann configuration over the w-plane [2, p. 130] onto which G is mapped by $w=f(z)$. Let the points $w_1, w_2, \cdots, w_m$ on R, all with affix $w=0$, represent respectively $f(a_1), f(a_2), \cdots, f(a_m)$. At each point w_i of the set $w_1, w_2, \cdots, w_m$ consider the radius $D_{k_i}(w_i)$ of k_i-valence there [2, pp. 161–162]. Let D_0 denote the greatest number to be found among the $D_{k_i}(w_i)$.

Any point w on the w-plane such that there is a point P of R whose affix is w will be said (as in [2]) to be covered by P. Let r_0 denote the radius of the largest circle K_0 of all circles K centered at $w=0$ such that every point within K is covered by at least point of R. On K_0 itself there will be at least one point which is not covered by any point of R. It is seen that $D_0 \leqq r_0$. Then we have the following appraisal of M.

THEOREM 1. $D_0 \leqq M \leqq r_0$.

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PROOF. Let $w=b$ be a point on K_0 which is not covered by any point of R. It follows at once that the function $F(z)=f(z)-b$ is in class S and such that $M(F)=|b|=r_0$. It remains to prove that $D_0\leqq M$. Let $g(z)$ be any function which is analytic interior to G and such that

$$|g(z)-f(z)| \leqq H < D_0, \qquad z \text{ in } G.$$

Cut through R with a circular biscuit cutter of radius C, where $H<C<D_0$, so that the centers of circular sheets thus cut from R have affix $w=0$. Let $z=q(w)$ denote the inverse of $w=f(z)$ and consider the transform under $z=q(w)$ of the k_h-sheeted circle [**2**, p. 159] of radius C inside the biscuit cutter with sheets hanging together at w_h, where w_h is a point of the set $w_1, w_2, \cdots, w_m$ such that $D_{k_h}(w_h)=D_0$. This transform is a simply connected region Q_h lying in G and bounded by a contour J_h also lying in G [**2**, p. 164]. On J_h we have $|f(z)|=C$ and $|g(z)-f(z)|<C$. It follows by Rouche's Theorem [**1**, p. 6] that the function $g(z)$ has precisely k_h zeros within J_h. Thus every function $g(z)$ for which l.u.b. $[|g(z)-f(z)|, z \text{ in } G]<D_0$ vanishes in G. Consequently M can not be less than D_0.

3. **Number of zeros of approximating functions.** By following through with the method employed in the proof of Theorem 1 we obtain a result on the number of zeros of approximating functions as follows. Let the distinct numbers to be found among the $D_{k_i}(w_i)$ arranged in order be $D_0>D_1>D_2> \cdots >D_p$. Then we have the following theorem.

THEOREM 2. *Let $f(z)$ be analytic and uniformly limited interior to a simply connected region G and have k zeros in G distributed as described above in* §2. *Let $g(z)$ be analytic interior to G and such that*

$$|g(z)-f(z)| \leqq C < D^*, \qquad z \text{ in } G,$$

where D^* is one of the numbers $D_0, D_1, D_2, \cdots, D_p$. Then $g(z)$ has at least as many zeros in G as the sum of the k_i for which $D_{k_i}(W_i)\geqq D^*$.

The proof of Theorem 2 is omitted, since it is essentially contained (except for the addition of the pertinent k_i which provide the count of the zeros of $g(z)$ within the contours J_i) in the latter part of the proof of Theorem 1.

We observe in particular that if

$$\text{l.u.b. } [|g(z)-f(z)|, z \text{ in } G] < D_p,$$

it follows at once by Theorem 2 that $g(z)$ has at least as many zeros in G as has $f(z)$. Indeed, we can make this observation precise, as shown in our next theorem.

THEOREM 3. *If $g(z)$ is analytic interior to G and such that*

$$|g(z) - f(z)| \leqq L < D_p, \qquad z \text{ in } G,$$

then $g(z)$ has precisely as many zeros in G as has $f(z)$.

PROOF. As we have already indicated, it follows by the method of proof used in Theorem 1 that $g(z)$ has precisely k zeros situated interior to the regions (bounded by the contours $J_1, J_2, \cdots, J_m$) which are the transforms by $z=q(w)$ of the m simply connected multi-sheeted circles (including single-sheeted circles, if any, corresponding to simple zeros of $f(z)$, if any) cut from R by a circular biscuit cutter of radius r, where $L<r<D_p$ and the centers of the circles all have affix $w=0$. If $g(z)$ were to have more than k zeros in G, all zeros other than the k zeros just mentioned would have to lie not interior to the contours $J_1, J_2, \cdots, J_m$. Suppose there were such an additional zero at $z=a$. Then the point $w=f(a)$ on R would not lie interior to the biscuit cutter; and we would have $|f(a)| \geqq r > L$. But $g(a)=0$. This would make $|g(a)-f(a)| > L$, contrary to hypothesis. Therefore $g(z)$ has precisely k zeros in G.

REFERENCES

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