ON TYPICALLY-REAL FUNCTIONS IN A CUT PLANE

BY E. P. MERKES

1. Introduction. Let

$$w = f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots \tag{1.1}$$

be regular in $|z| < 1$ and real valued if and only if z is real and $-1 < z < 1$. Then $f(z)$ is said to be typically-real in the unit circle. Rogosinski [**3**] has shown that a necessary and sufficient condition for a regular function $f(z)$ in $|z| < 1$, where $f(0) = 0$, $f'(0) = 1$, to be typically-real is that

$$f(z) = \frac{z}{1 - z^2} \phi(z), \qquad \phi(0) = 1, \tag{1.2}$$

where $\phi(z)$ is regular, real for real z, and has positive real part for $|z| < 1$. Furthermore, if $w = f(z)$ maps each circle $|z| = r < 1$ into a contour having the property that every line parallel to the imaginary axis cuts this contour in at most two points, then $f(z)$ is said to be convex in the direction of the imaginary axis. A necessary and sufficient condition that $f(z)$ be convex in the direction of the imaginary axis is that $zf'(z)$ be typically-real in the unit circle [**1**; **2**].

It is the main purpose of this paper to display a connection between typically-real functions on one hand and Stieltjes transforms and continued fractions on the other. To this end consider first a function $F(\zeta)$ which is real valued for real ζ and regular in the complex plane cut along the negative real axis from $-\infty$ to -1. If, further, $F(0) = 0$, $F'(0) = 1$, and $\operatorname{Im} F(\zeta) \neq 0$ for nonreal values of ζ in this cut plane, the function $F(\zeta)$ is said to be typically-real in the cut plane. The class of all such functions $F(\zeta)$ is denoted by $T[-\infty, -1]$. On the other hand, if $F(0) = 1$ and $\operatorname{Re} F(\zeta) > 0$ for ζ in this cut plane, we say that $F(\zeta)$ is in the class $R[-\infty, -1]$.

If the usual agreements are made regarding the normalization, a similar definition can be given for a function to be typically-real in the complex plane cut along the real axis from a to b such that the cut does not include both 0 and ∞. In each case there is a linear fractional transformation with real coefficients which carries such a function into one which, except for the normalization, is in the class $T[-\infty, -1]$. For this reason attention is confined to the latter class.

Presented to the Society, January 22, 1959; received by the editors November 28, 1958 and, in revised form, February 19, 1959.

2. **Some characterizations.** The univalent transformation

$$\zeta = \frac{4z}{(1-z)^2} \tag{2.1}$$

maps the disc $|z| < 1$ onto the ζ-plane cut along the negative real axis from ∞ to -1. Since $\operatorname{Im} \zeta = 4(1-|z|^2)^{-1} \operatorname{Im} z$, the function

$$f(z) = \frac{1}{4} F\left[\frac{4z}{(1-z)^2}\right] \tag{2.2}$$

is typically-real in the unit circle whenever $F(\zeta)$ is in the class $T[-\infty, -1]$ and conversely. Thus there is a one-to-one correspondence between the class $T[-\infty, -1]$ and the class of typically-real functions in the unit circle.

By (1.2) and (2.2) we obtain, after the change of variable (2.1), the following restatement of the cited theorem of Rogosinski:

THEOREM 2.1. *$F(\zeta)$ is in $T[-\infty, -1]$ if and only if there exists a function $\Phi(\zeta)$ in the class $R[-\infty, -1]$ such that*

$$F(\zeta) = \frac{\zeta}{(1+\zeta)^{1/2}} \Phi(\zeta), \tag{2.3}$$

where that branch of $(1+\zeta)^{1/2}$ is chosen in the cut plane which is positive at $\zeta = 0$.

It is known [6, p. 278] that $\Phi(\zeta)$ is in $R[-\infty, -1]$ if and only if there exists a nondecreasing function $\alpha(t)$ on $0 \leqq t \leqq 1$ such that $\alpha(1) - \alpha(0) = 1$ and

$$\Phi(\zeta) = (1+\zeta)^{1/2} \int_0^1 \frac{d\alpha(t)}{1+\zeta t}\cdot \tag{2.4}$$

Thus we have the following result:

COROLLARY 2.1. *A necessary and sufficient condition for $F(\zeta)$ to be in $T[-\infty, -1]$ is that there exists a nondecreasing function $\alpha(t)$ on $0 \leqq t \leqq 1$ such that $\alpha(1) - \alpha(0) = 1$ and*

$$F(\zeta) = \int_0^1 \frac{\zeta d\alpha(t)}{1+\zeta t}\cdot \tag{2.5}$$

By well-known [6, p. 263] relations between the Hausdorff moment problem and S-fractions, the following corollaries are obtained from the above:

COROLLARY 2.2. *$F(\zeta)$ is in $T[-\infty, -1]$ if and only if*

$$F(\zeta) = \frac{\zeta}{1+}\,\frac{(1-g_0)g_1\zeta}{1}\,{+}\cdots{+}\,\frac{(1-g_n)g_{n+1}\zeta}{1}\,{+}\cdots, \tag{2.6}$$

where $0 \leqq g_n \leqq 1$, $n=0, 1, 2, \cdots$.

COROLLARY 2.3. *Let*

$$F(\zeta) = \zeta - \alpha_2\zeta^2 + \alpha_3\zeta^3 - \cdots + (-1)^{n+1}\alpha_n\zeta^n + \cdots, \qquad |\zeta| < 1. \tag{2.7}$$

Then $F(\zeta)$ is in $T[-\infty, -1]$ if and only if the sequence $1, \alpha_2, \alpha_3, \cdots, \alpha_n, \cdots$ is totally monotone.

By (2.1), (2.2), and (2.6) we obtain, after an equivalence transformation, the following continued fraction characterization of this class:

THEOREM 2.2. *A necessary and sufficient condition for $f(z)$ to be typically-real in the unit circle is that $f(z)$ have a continued fraction expansion of the form*

$$f(z) = \frac{z}{1-z}\left[\frac{1}{1-z+}\,\frac{4(1-g_0)g_1z}{1-z}\,{+}\cdots{+}\,\frac{4(1-g_n)g_{n+1}z}{1-z}\,{+}\cdots\right], \tag{2.8}$$

where $0 \leqq g_n \leqq 1$, $n=0, 1, 2, \cdots$.

The correspondence between the classes under consideration can also be used to characterize typically-real functions in $|z| < 1$ in terms of Schur summability. For this purpose let $g(z) = \sum_{j=0}^{\infty} c_j(-z)^j$, $G(\zeta) = \sum_{j=0}^{\infty} \gamma_j(-\zeta)^j$ be related by

$$\frac{1}{2}(1-z)g(z) = G(\zeta), \qquad \zeta = 4z/(1-z)^2. \tag{2.9}$$

The transformation from the sequence $s_n = \sum_{j=0}^{n} c_j$ to the sequence $S_n = \sum_{j=0}^{n} \gamma_j$ by means of the identity (2.9) is called the (S)-transformation. The sequence $\{s_n\}$ is said to be (S)-summable to the value S if $S = \lim_{n\to\infty} S_n$. Scott and Wall [4] have introduced and studied in detail this consistent method of summability.

THEOREM 2.3. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $a_0 = 0$, $a_1 = 1$, be regular and real for real z in the unit circle. Then $f(z)$ is typically-real in $|z| < 1$ if* *and only if the sequence* $\{1+(-1)^n(a_{n+2}-a_n)/2\}_{n=0}^{\infty}$ *is* (S)*-summable to a value not exceeding unity and such that* $\{S_n-S_{n-1}\}_{n=0}^{\infty}$, *where* $S_{-1}=0$ *and the sequence* S_n *is the* (S)*-transform of the given sequence, is totally monotone.*

PROOF. By (2.4) and the characterization of Rogosinski $f(z)$ is typically-real in the unit circle if and only if

$$\begin{aligned} &\frac{(1-z)^2}{4z}\left[\frac{(1+z)^2}{z}f(z)-1\right]=\frac{(1-z)^2}{4z}\left[\frac{1+z}{1-z}\phi(z)-1\right] \\ (2.10)\qquad &=\frac{1}{\zeta}\left[(1+\zeta)^{1/2}\Phi(\zeta)-1\right]=\frac{1}{\zeta}\left[(1+\zeta)\int_0^1\frac{d\alpha(t)}{1+\zeta t}-1\right] \\ &=\int_0^1\frac{(1-t)d\alpha(t)}{1+\zeta t}, \end{aligned}$$

where $\zeta=4z/(1-z)^2$, $\Phi(\zeta)=\phi(z)$, and $\alpha(1)-\alpha(0)=1$. Let $G(\zeta)$ be the function represented by the integral in (2.10). Then $\{\gamma_j\}$ is totally monotone and $\sum_{j=0}^{\infty}\gamma_j\leqq 1$ where $G(\zeta)=\sum_{j=0}^{\infty}\gamma_j(-\zeta)^j$ for $|\zeta|<1$ [**6**, p. 284]. Moreover, if the left-hand side of (2.10) is taken to be the function $(1-z)g(z)/2$ in (2.9), it is easily seen that $s_n=\sum_{j=0}^{n}c_j$ $=1+(-1)^n(a_{n+2}-a_n)/2$. The theorem now follows at once from the definition of (S)-summability.

3. **Some properties of the class** $T[-\infty, -1]$. Among other considerations the following theorem yields some properties of S-fractions of the form (2.6):

THEOREM 3.1. *If* $F(\zeta)$ *is in* $T[-\infty, -1]$, *then it is univalent for* $\operatorname{Re}\zeta>-1$. *This result is sharp. Moreover,* $F(\zeta)$ *is convex in the direction of the imaginary axis for* $|\zeta|<1$.

The domain of univalence, $\operatorname{Re}\zeta>-1$, for the class is obtained by a trivial adjustment of a proof given by Thale [**5**, p. 233]. That the result is sharp is established by considering the following functions of $T[-\infty, -1]$:

$$F(\zeta;c)=\frac{\zeta(1+c\zeta)}{1+\zeta},\qquad 0<c<1. \tag{3.1}$$

It is easily seen that for any two distinct points ζ_1 and ζ_2 in the upper (lower) half-plane and on the line $\operatorname{Re}\zeta=-1$ there is a constant c_0, $0<c_0<1$, such that $F(\zeta_1;c_0)=F(\zeta_2;c_0)$.

In order to prove that $F(\zeta)$ is convex in the direction of the imaginary axis, note that by (2.5)

$$\text{(3.2)} \qquad \operatorname{Im}\left[\zeta F'(\zeta)\right] = \operatorname{Im} \zeta \int_0^1 \frac{1 - |\zeta|^2 t^2}{|1 + \zeta t|^4}\, d\alpha(t).$$

It immediately follows that $\zeta F'(\zeta)$ is typically-real. This yields the desired result.

It is interesting to note [5] that the domain of univalence for the class $T[-\infty, -1]$ in Theorem 3.1 is also a sharp domain of univalence for functions of the form $F(\zeta)/\zeta$, where $F(\zeta)(\not\equiv\zeta)$ is in $T[-\infty, -1]$. One way to see this is to observe that by (2.7)

$$\text{(3.3)} \qquad F(\zeta)/\zeta = 1 - \alpha_2 F_1(\zeta).$$

By a property of totally monotone sequences it now follows from Corollary 2.3 that $F_1(\zeta)$ is in $T[-\infty, -1]$. The univalence of $F_1(\zeta)$ then implies that of $F(\zeta)/\zeta$.

4. **A domain of univalence for typically-real functions in the unit circle.** Let $f(z)$ be typically-real in the unit circle. By the one-to-one correspondence of §2 there exists a function $F(\zeta)$ in $T[-\infty, -1]$ such that (2.2) holds. From Theorem 3.1 and the fact that (2.1) is a univalent transformation, we conclude that $f(z)$ is univalent for

$$\text{(4.1)} \qquad \operatorname{Re}\left[\frac{4z}{(1-z)^2}\right] > -1.$$

The region (4.1) can be expressed in the form given in the following result:

THEOREM 4.1. *If $f(z)$ is typically-real in the unit circle, then $f(z)$ is univalent for z in the domain D bounded by the circular arcs $z = re^{i\theta}$, where*

$$\text{(4.2)} \qquad r = (1 + \sin^2\theta)^{1/2} - |\sin\theta|, \qquad 0 \leqq \theta < 2\pi.$$

The result is sharp in the sense that any open region of univalence for this class of functions which contains D is coincident with D.

The sharpness result of the theorem follows from that for the class $T[-\infty, -1]$.

The largest circular domain with center at the origin and contained in the domain D has radius $(2)^{1/2} - 1$.

COROLLARY 4.1. *If $f(z)$ is typically-real in the unit circle, then $f(z)$ is univalent in the disc $|z| < (2)^{1/2} - 1$. This circular domain of univalence cannot be replaced by a larger circular domain with center at the origin in which each function of the class under consideration is univalent.*

The function

$$f(z) = z(1+z^2)/(1-z^2)^2 \tag{4.3}$$

is typically-real in the unit circle. Moreover, since the derivative of this function vanishes at $z = \pm((2)^{1/2}-1)i$, (4.3) is not univalent in any larger circular domain with center at the origin than that given in Corollary 4.1.

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ON THE POLE AND ZERO LOCATIONS OF RATIONAL LAPLACE TRANSFORMATIONS OF NON-NEGATIVE FUNCTIONS

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Let $a(t)$ be a real, bounded function of the real variable t defined in the interval, $0 \leqq t < \infty$, and let its Laplace-Stieltjes transform,

$$F(s) = \int_0^\infty e^{-st} da(t), \tag{1}$$

be a rational function of the complex variable $s = \sigma + i\omega$, having at least as many poles as zeros. $F(s)$ may be written as

$$F(s) = \frac{\prod_{i=1}^{h} (s-\eta_i) \prod_{i=1}^{g} (s-\nu_i)}{\prod_{i=1}^{m} (s-\rho_i) \prod_{i=1}^{q} (s-\xi_i)} \tag{2}$$

where the η_i and ρ_i are real and the ν_i and ξ_i are complex. Under these

Received by the editors December 22, 1958 and, in revised form, February 10, 1959.