SOLUTIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS WHICH ARE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER[1]

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Let $\mathfrak{F}$ be an ordinary differential field (i.e., a field with a given derivation) of characteristic zero. An element z belonging to a differential field extension of $\mathfrak{F}$ is said to be of order r over $\mathfrak{F}$ if the lowest order irreducible differential equation with coefficients in $\mathfrak{F}$ that z satisfies is of order r. It follows that z is of order r over $\mathfrak{F}$ if and only if the degree of transcendency of $\mathfrak{F}\langle z\rangle$ over $\mathfrak{F}$ is r.

In [**4**] Ritt proved that if P, $Q\in\mathfrak{F}\{y\}$ and Q vanishes for every zero of P then the sum of the lowest (highest) degree terms of Q vanishes for every zero of the sum of the lowest (highest) degree terms of P.

For our purpose we need the following slight generalization; it can be obtained either by essentially the same proof as used by Ritt, or as an almost immediate corollary of this theorem.

THEOREM 1. *Let P, $Q\in\mathfrak{F}\{y\}$ and let Q vanish for every zero of P which is of order $\geqq r$ over $\mathfrak{F}$; then the sum of lowest (highest) degree terms of Q vanishes for every zero of the sum of lowest (highest) degree terms of P which is of order $\geqq r$ over $\mathfrak{F}$.*

THEOREM 2. *Let z be a zero of an nth order linear differential polynomial $L(y)\in\mathfrak{F}\{y\}$ and let the order of z over $\mathfrak{F}$ be 1. Then there exists an integer r, $0\leqq r<n$, such that $z^{(r)}$ is a zero of an irreducible first order differential polynomial $P(y)\in\mathfrak{F}\{y\}$ the sum of whose highest degree terms is of order 1 and $z^{(r)}$ is of order 1 over $\mathfrak{F}$.*

PROOF. Since z is of order 1 over $\mathfrak{F}$, $\mathfrak{F}\langle z\rangle=\mathfrak{F}(z, z')$, which is an algebraic function field of one variable over $\mathfrak{F}$. Let v be an infinite valuation on $\mathfrak{F}(z, z')$ such that v is trivial on $\mathfrak{F}$ and for any $Q(z)$, $R(z)\in\mathfrak{F}[z]v(Q/R)=$degree of $R-$degree of Q. Since z is a zero of an nth order linear differential polynomial we can not have $v(z^{(s+1)})<v(z^{(s)})$ for all s less than n. Let r be the smallest integer such that $v(z^{(r+1)})\geqq v(z^{(r)})$; then the order of $z^{(r)}$ over $\mathfrak{F}$ is 1. For, $z^{(r)}\in\mathfrak{F}\langle z\rangle$ so that the order of $z^{(r)}$ over $\mathfrak{F}$ is $\leqq 1$; if $z^{(r)}$ were algebraic over $\mathfrak{F}$ then $v(z^{(r)})$ would be zero, which is greater than $v(z)$, contradicting our assumption on the minimality of r. Let $P(y)\in\mathfrak{F}\{y\}$ be the first order ir-

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reducible differential polynomial which vanishes for $z^{(r)}$. Since at least two terms of $P(z^{(r)})$ must have the same smallest value under the given valuation v, it follows that the highest degree terms of $P(y)$ are of order 1.

A homogeneous linear differential polynomial $L(y)\in\mathfrak{F}\{y\}$ is said to be linearly reducible over $\mathfrak{F}$ if there exist homogeneous linear differential polynomials $M(y)$, $N(y)\in\mathfrak{F}\{y\}$, each one of positive order, such that

$$L(y) = M(N(y)).$$

If no such decomposition exists we say $L(y)$ is linearly irreducible over $\mathfrak{F}$.

THEOREM 3. *Let $L(y)\in\mathfrak{F}\{y\}$ be a homogeneous linear differential polynomial linearly irreducible over $\mathfrak{F}$. If a zero z of $L(y)$ is of order* 1 *over $\mathfrak{F}$, then there exists a fundamental system of zeros $(u_1, \cdots, u_n)$ of $L(y)$ such that u_i'/u_i, $i=1, \cdots, n$, is algebraic over $\mathfrak{F}$.*

PROOF. Since $L(y)$ is linearly irreducible over $\mathfrak{F}$ it suffices[2] to show the existence of one zero u of $L(y)$ such that u'/u is algebraic over $\mathfrak{F}$ [**2**]. By Theorem 2 there exists an integer r such that $z^{(r)}$ is a zero of a first order differential polynomial $P(y)\in\mathfrak{F}\{y\}$ the sum of whose highest degree terms is of order 1. Because $L(y)$ is linearly irreducible there exist homogeneous linear differential polynomials $M(y)$, $N(y)\in\mathfrak{F}\{y\}$, $M(y)$ linearly irreducible over $\mathfrak{F}$, such that $M(z^{(r)})=0$ and, for any nontrivial zero w of $M(y)$, $N(w)$ is a nontrivial zero of $L(y)$ [**5**, vol. 2, pp. 164, 165]. By Theorem 1 there exists a nontrivial zero w of the sum of the highest degree terms of $P(y)$ such $M(w)=0$. Since w is a zero of a homogeneous first order differential polynomial w'/w is algebraic over $\mathfrak{F}$. Let $u=N(w)$ then $L(u)=0$. Since w'/w is algebraic over $\mathfrak{F}$, $u=kw$, where k belongs to an algebraic extension of $\mathfrak{F}$, so that u'/u is algebraic over $\mathfrak{F}$ and our theorem follows.

DEFINITION. If the lowest order *linear* differential polynomial which vanishes for z is of order n, then we say that the *linear order* of z over $\mathfrak{F}$ is n. Let the linear order of z over $\mathfrak{F}$ be n and let V be the set of all linear differential polynomials in z with coefficients in $\mathfrak{F}$. V is, in an obvious way, an $n+1$ dimensional vector space over $\mathfrak{F}$ so that the linear order of any element of V over $\mathfrak{F}$ is $\leqq n$.

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[2] Loewy assumes that $\mathfrak{F}$ has an algebraically closed field of constants and uses in his proof the automorphisms of a Picard-Vessiot extension of $\mathfrak{F}$. It is easily seen, however, that by the substitution of relative isomorphisms over $\mathfrak{F}$ for his automorphisms, the proof remains valid. For, we can use the theorem by Kolchin [1] that an element t belonging to $\mathfrak{K}$, a differential field extension of $\mathfrak{F}$, which is left invariant by every isomorphism of $\mathfrak{K}$ over $\mathfrak{F}$, belongs to $\mathfrak{F}$.

THEOREM 4. *Let z be a zero of a first order differential polynomial $P(y) \in \mathfrak{F}\{y\}$ and of a linear differential polynomial $L(y) \in \mathfrak{F}\{y\}$, and let $\bar{\mathfrak{F}}$ be the algebraic closure of $\mathfrak{F}$. There exists $u \in \bar{\mathfrak{F}}\langle z\rangle$ such that z is algebraic over $\mathfrak{F}\langle u\rangle$ and the linear order of u over $\bar{\mathfrak{F}}$ is* 1.

REMARK. If a first order differential polynomial $P(y) \in \mathfrak{F}\{y\}$ factors over $\bar{\mathfrak{F}}$ into linear factors, it is well known that any zero u of $P(y)$ is a zero of a linear differential polynomial of higher order with coefficients in $\mathfrak{F}$ [**3**]. Also, any polynomial $z \in \bar{\mathfrak{F}}\{u\}$ is a zero of a first order differential polynomial $Q(y) \in \mathfrak{F}\{y\}$ and of a linear differential polynomial with coefficients in $\mathfrak{F}$. $Q(y)$ may remain irreducible over $\bar{\mathfrak{F}}$. For example, let $\mathfrak{F}$ be the field of rational numbers, $u = e^x$, $z = e^x + e^{2x}$; $Q(y) = (2y - y')^2 - (y' - y)$ is, obviously, irreducible over $\bar{\mathfrak{F}}$. Our theorem states that besides the obvious cases just mentioned there is only one more possibility for an element z to be simultaneously a zero of a first order and of a linear differential polynomial; namely that z belongs to an algebraic extension of $\bar{\mathfrak{F}}\langle u\rangle$ where u is a zero of a first order linear differential polynomial with coefficients in $\bar{\mathfrak{F}}$.

PROOF OF THEOREM. If $z \in \bar{\mathfrak{F}}$ we take $u = z$. Let the order of z over $\mathfrak{F}$ (and hence over $\bar{\mathfrak{F}}$) be 1. Let V be the vector space of all linear differential polynomials in z with coefficients in $\bar{\mathfrak{F}}$. Since z is a zero of a linear differential polynomial, V is a finite dimensional vector space. For any element $v \in V$, $\bar{\mathfrak{F}}\langle v\rangle \subseteq \bar{\mathfrak{F}}\langle z\rangle$ so that order of v over $\bar{\mathfrak{F}}$ is $\leqq 1$. Let A be the set of all elements v in V such that order of v over $\bar{\mathfrak{F}}$ is 1. A is not empty since $z \in A$. Of all the elements in A choose u such that the linear order of u over $\bar{\mathfrak{F}}$ is least. We are going to show that the linear order of u over $\bar{\mathfrak{F}}$ is 1.

Let the linear order of u over $\bar{\mathfrak{F}}$ be n and let W be the $n+1$ dimensional vector space over $\bar{\mathfrak{F}}$ of all linear differential polynomials in u with coefficients in $\bar{\mathfrak{F}}$. For any $w \in W - \bar{\mathfrak{F}}$ the following holds:

(1) $w \in A$ (i.e. w is of order 1 over $\bar{\mathfrak{F}}$).

(2) Linear order of w over $\bar{\mathfrak{F}}$ is n.

(3) If the nth order linear equation that w satisfies over $\bar{\mathfrak{F}}$ is $M(y) = f$, $M(y)$ homogeneous, $f \in \bar{\mathfrak{F}}$; then $M(y)$ is linearly irreducible.

To prove (1) note that $w \in V - \bar{\mathfrak{F}}$ and, since $\bar{\mathfrak{F}}$ is algebraically closed the order of w over $\bar{\mathfrak{F}}$ is 1. Since $w \in A$ the linear order of w over $\bar{\mathfrak{F}}$ is $\geqq n$ (since n was least) but $w \in W$ and each element in W has linear order over $\bar{\mathfrak{F}} \leqq n$. This proves (2). To prove (3) we note that if $M(y) = N_2(N_1(y))$, $N_1(y)$ of positive order, then the linear order of $N_1(w)$

over $\bar{\mathfrak{F}}$ is the order of N_2 which is less than linear order of w over $\bar{\mathfrak{F}}$ contradicting (2). Hence $M(y)$ is linearly irreducible over $\bar{\mathfrak{F}}$.

Now, by Theorem 2, there exists $w \in W - \bar{\mathfrak{F}}$ such that w is a zero of a first order differential polynomial $Q(y) \in \bar{\mathfrak{F}}\{y\}$ the sum of whose highest degree terms is of order 1. Since $\bar{\mathfrak{F}}$ is algebraically closed the sum of the highest degree terms factors into linear factors with at least one of the factors $N_1(y)$ of order 1. By Theorem 1 a generic zero of the prime differential ideal generated by $N_1(y)$ is a zero of $M(y)$ (since the generic zero is of order 1 over $\bar{\mathfrak{F}}$), so that $M(y)$ belongs to the prime differential ideal $\{N_1(y)\}$. Since $M(y)$ is linear (i.e. of the same degree as $N(y)$), $M(y) = N_2(N_1(y))$. By (3) $M(y)$ is linearly irreducible so that $N_2(y)$ is of order zero and $M(y)$ is of order 1. By (2) this implies that $n = 1$ and the linear order of u over $\bar{\mathfrak{F}}$ is 1. Since u is of order 1 over $\mathfrak{F}$ it follows that z is algebraic over $\mathfrak{F}\langle u\rangle$; this proves our theorem.

COROLLARY. *If a zero z of a first order differential polynomial $P(y) \in \mathfrak{F}\{y\}$ is a zero of a linear differential polynomial $L(y) \in \mathfrak{F}\{y\}$, then either z is algebraic over $\mathfrak{F}$ or $P(y)$ is solvable by quadratures.*

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