ON A PROBLEM OF WOLK IN INTERVAL TOPOLOGIES

T. NAITO

1. **Introduction.** Let P be a partially ordered set (poset) with respect to a relation $\leqq$. We say that two elements x and y in P are incomparable if and only if $x \nleqq y$ and $x \ngeqq y$. Let us call a subset M of P diverse if and only if $x \in M$, $y \in M$ and $x \neq y$ imply that x and y are incomparable. We define the width of P to be l.u.b. $\{k \mid k$ is the cardinal number of a diverse subset of $P\}$.

We shall call a subset M of P Dedekind-closed if and only if whenever D is an up-directed subset of M and $y =$ l.u.b. D or D is a down-directed subset of M and $y =$ g.l.b. D, we have $y \in M$. We define a topology $\mathfrak{D}$ on P whose closed sets are precisely the Dedekind-closed subset of P and let $\mathfrak{g}$ denote the interval topology on P, which is obtained by taking all sets of the form $[a, b]$ as a sub-basis for the closed sets.

E. S. Wolk introduced the following concept [1]:

DEFINITION. If $\mathfrak{J}$ is a topology defined on P, we shall say that $\mathfrak{J}$ is order-compatible with P if and only if

(i) every set closed with respect to $\mathfrak{J}$ is Dedekind-closed, and

(ii) every set of the form $\{x \in P \mid a \leqq x \leqq b\}$ is closed with respect to $\mathfrak{J}$.

He proved the following theorem in his paper [1].

THEOREM. *If P is a poset of finite width, then P possesses a unique order-compatible topology.*

And he proposed the question: "Whether, in the above theorem, the hypothesis that P is of finite width, can be replaced by the weaker condition that P contains no infinite diverse subset."

The main purpose of this note is to give the answer to the above question, and it is contained in the following theorems.

THEOREM 1. *If P contains no infinite diverse set, then P possesses a unique order-compatible topology.*

THEOREM 2. *Let P be a complete lattice. Then P possesses a unique order-compatible topology if and only if P contains no infinite diverse set.*

2. **Main theorems.** First we shall prove the following lemma which is the main result in this paper.

Received by the editors February 18, 1959.

LEMMA. *Let P be a poset containing no infinite diverse set and f be a net on A with range $(f)=S\subset P$. If element y of P is the l.u.b. of the range of every subnet of f, then there exists an up-directed set $M\subset S$ such that $y=$l.u.b. (M).*

PROOF. Let us suppose that the lemma is false. Let M_1 be any maximal up-directed subset of S (which exists by Zorn's lemma). By the assumption that the lemma is false, we have $y\neq$l.u.b. M_1. Hence there exists no subnet of f with range contained in M_1. Therefore there exists $\alpha_1\in A$ such that $f(\alpha)\in S-M_1$ for all $\alpha\geqq\alpha_1$. Next let us choose a maximal up-directed subset M_2 of $\{f(\alpha)\mid\alpha\geqq\alpha_1\}$. Then we have $y\neq$l.u.b. M_2 and there exists $\alpha_2\in A$ such that $f(\alpha)\in S-M_1-M_2$ for all $\alpha\geqq\alpha_2\geqq\alpha_1$. Now choose M_3, a maximal up-directed subset of $\{f(\alpha)\mid\alpha\geqq\alpha_2\}$, and continue the above process.

Thus we obtain a countable infinite set of maximal up-directed subsets: $M_1, M_2, M_3, \cdots$. From the fact that M_i is a maximal up-directed set, we have

(*) $x\in M_1$, $y\in M_2$ imply $x\ngeqq y$. More generally, $x\in M_i$, $y\in M_j$, $i<j$ imply $x\ngeqq y$.

If for all pairs $x\in\bigcup_i M_i$, $y\in\bigcup_i M_i$, there is an element z of $\bigcup_i M_i$ such that $x\leqq z$, $y\leqq z$, then $\bigcup_i M_i$ is an up-directed set which contradicts the fact that each M_i is a maximal up-directed set. Therefore, there exist elements $a_1\in M_k$ and $b_1\in M_l$ such that $\bigcup_i M_i$ contains no z such as $z\geqq a_1$ and $z\geqq b_1$. By the definition of a_1 and b_1 there exists either an infinite number of M_i which contains no upper bound of a_1 or an infinite number of M_i which contains no upper bound of b_1. In fact, if there only exists a finite number of M_i which contains no upper bound of a_1 and M_j which contains upper bound of b_1, then there exists M_n containing an upper bound of a_1 and b_1 which contradicts the definition of a_1 and b_1. For example, if there exists an infinite number of M_i (max, $(k, 1)\leqq i$) which contains no upper bound of b_1, then we put $c_1=b_1$ and denote such M_i by $M_1^2, M_2^2, M_3^2, \cdots$ (in the same order as M_i).

Similarly, there exist elements $a_2\in M_{k'}^2$ and $b_2\in M_{l'}^2$ such that $\bigcup_i M_i^2$ does not contain z such as $z\geqq a_2$, $z\geqq b_2$. By the definition of a_2 and b_2, there exists either an infinite number of M_i^2 which contains no upper bound of a_2 or an infinite number of M_i^2 which contains no upper bound of b_2. For example, if there exists an infinite number of M_i^2 (max $(k', l')\leqq i$) which contains no upper bound of a_2, then we put $c_2=a_2$ and denote such M_i^2 by $M_1^3, M_2^3, M_3^3, \cdots$ (in the same order as M_i).

Continuing this process, we have an infinite set $c_1, c_2, \cdots$. Set

$\{c_i\}$ is an infinite diverse set of P. In fact, by the definition of M_i, $x \in M_i^n$, $y \in M_k^n$ and $i<k$ imply $x \ngeqq y$. Hence by the definition of c_k we have $c_i \ngeqq c_k$ for $i<k$. Since each of M_j^n contains no upper bound of c_{n-1}, we have $c_i \nleqq c_k$ for $i<k$. Therefore we have $c_i \nleqq c_k$ for $i \neq k$. The proof is complete.

We obtain from the above lemma the following theorem.

THEOREM 1. *If P contains no infinite diverse set, then P possesses a unique order-compatible topology.*

The proof will not be given since it is exactly the same as for the proof of Wolk's Theorem 1 in [1].

THEOREM 2. *Let P be a complete lattice. Then, P possesses a unique order-compatible topology if and only if P contains no infinite diverse set.*

PROOF. Since P is a complete lattice, P is compact in the interval topology [2]. Now, suppose that P possesses a unique order-compatible topology, then P is compact in the $\mathfrak{D}$-topology. Suppose that $\{a_i \mid i=1, 2, \cdots\}$ is an infinite diverse subset of P. Let $F_n = \{a_i \mid i \geqq n\}$. Then F_n is closed in the $\mathfrak{D}$-topology and the family of all F_n has the finite intersection property. But $\bigcap F_n$ is empty which is a contradiction.

Since the necessity of the condition is Theorem 1, then the theorem is proved.

REFERENCES

1. E. S. Wolk, *Topologies on a partially ordered set*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 524–529.

2. O. Frink, *Topology in lattices*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 569–582.

YAMANASHI UNIVERSITY, KOFUCHUMACHI, KOFUSHI, JAPAN