MODULARITY RELATIONS IN LATTICES

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1. **Introduction.** Linear independence has been formulated lattice-theoretically by G. Birkhoff [1], J. von Neumann [4] and, in particular, L. R. Wilcox [5], who studied it in connection with ordinary modularity considered as a binary relation. In this work, the concept of a modularity relation is defined abstractly from which the theory of independence is developed. These results generalize those of S. Maeda [2] whose abstraction of independence characterizes ordinary independence. Also quasi-modularity relations are considered abstractly, which relations arise in the theory of quasi-dual-ideals [7]. Relations studied earlier by the author [3] are shown to be instances of the abstract relations considered here.

Throughout this paper L is to be a lattice with order $\leqq$, join $+$ and meet $\cdot$. For $b, c \in L$, $(b, c)M$ (read (b, c) *modular*) means $(a+b)c = a+bc$ for every $a \leqq c$ (M will be referred to as ordinary modularity).

The notations $\subset$, $+$, $\cdot$, Θ, $\times$ are respectively set-theoretic inclusion, sum, product, the empty set and cartesian product, and the set of all elements x with the property $E(x)$ is denoted by $[x; E(x)]$.

2. **Modularity relations and independence.** First, the notion of a modularity relation is defined abstractly, which is then used in the definition of the independence relation and the development of the independence theory.

(2.1) Definition. Let $R \subset T \subset L \times L$. The relation R is a *modularity relation under* T means

(a) $(b, c)R$, $b' \leqq b$, $c' \leqq c$, $b'c' = bc$, $(b', c')T$ implies $(b', c')R$;

(b) $(c, d)R$, $(b, c+d)R$, $b(c+d) = cd$ implies $(b+c, d)R$, $(b+c)d = cd$.

Part (a) of the definition would be too broad for the purposes considered here if the condition $(b', c')T$ were omitted from the hypotheses. The set T is introduced merely to provide a control on the pairs that are eligible to be in R and its role will become evident in the examples considered in the subsequent sections.

(2.2) Definition. For R a modularity relation under T, R is said

(a) to satisfy the *intersection property* if $(c, d)R$, $(b, c+d)R$, $b(c+d) = cd$ implies $(b+d)(c+d) = d$;

(b) to be *symmetric at* a, for $a \in L$, if $(b, c)R$, $bc = a$ implies $(c, b)R$.

Examples exist showing that a modularity relation does not necessarily satisfy these properties.

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(2.3) Definition. Let R be a modularity relation under T. For $n \geqq 2$, $a, a_1, \cdots, a_n \in L$, $(a_1, \cdots, a_n)R_a$ (read $(a_1, \cdots, a_n)$ *R-independent over* a) means $(\sum_U a_i, \sum_V a_i)R$, $(\sum_U a_i)(\sum_V a_i) = a$ for every nonempty $U, V \subset [1, \cdots, n]$ such that $j < k$ for $j \in U$, $k \in V$.

Throughout this section it is assumed that R is a modularity relation under T, $n \geqq 2$ and $a, a_1, \cdots, a_n \in L$.

(2.4) Corollary. *Let* $(a_1, \cdots, a_n)R_a$.

(a) *If* $a_i \neq a$ *for* $1 \leqq i \leqq n$, *then* $a_i \neq a_j$ *for* $i \neq j$.

(b) *If* $1 \leqq k_1 < \cdots < k_m \leqq n$, $m \geqq 2$, *then* $(a_{k_1}, \cdots, a_{k_m})R_a$.

(c) *If* $a \leqq a_i' \leqq a_i$ *for* $1 \leqq i \leqq n$, *then* $(a_1', \cdots, a_n')R_a$ *provided* $(\sum_U a_i', \sum_V a_i')T$ *for every nonempty* $U, V \subset [1, \cdots, n]$ *such that* $j < k$ *for* $j \in U$, $k \in V$.

(2.5) Theorem. *If* $(a_1, \cdots, a_n)R_a$, *then* $(a_i, a_{i+1} + \cdots + a_n)R_a$ *for every* $i = 1, \cdots, n-1$, *and conversely, provided* $(a_i, \sum_V a_j)T$ *for every nonempty* $V \subset [i+1, \cdots, n]$.

Proof. The forward implication is immediate. The reverse is obvious for $n = 2$. Suppose it holds for $q \leqq n-1$ where $n \geqq 3$. Let $(a_i, a_{i+1} + \cdots + a_n)R_a$ for $i = 1, \cdots, n-1$ and let $U, V \subset [1, \cdots, n]$ such that U, V are nonempty and $j < k$ for $j \in U$, $k \in V$. Denote U by $[j_1, \cdots, j_u]$ and V by $[k_1, \cdots, k_v]$, where, without loss of generality, $j_1 < \cdots < j_u < k_1 < \cdots < k_v$. Then by (2.4.c),

$$(a_{j_i}, a_{j_{i+1}} + \cdots + a_{k_v})R_a \qquad \text{for } i = 1, \cdots, u$$

and

$$(a_{k_i}, a_{k_{i+1}} + \cdots + a_{k_v})R_a \qquad \text{for } i = 1, \cdots, v-1.$$

In case $U + V \neq [1, \cdots, n]$, it follows from the induction hypothesis that $(a_{j_1}, \cdots, a_{j_u}, a_{k_1}, \cdots, a_{k_v})R_a$, whence $(\sum_U a_i, \sum_V a_i)R_a$. Let $U + V = [1, \cdots, n]$. From the above argument,

$$\left(a_{j_2} + \cdots + a_{j_u}, \sum_V a_i\right) R_a,$$

and by hypotheses, $(a_{j_1}, a_{j_2} + \cdots + a_{j_u} + \sum_V a_i)R_a$. Thus (2.1.b) yields $(\sum_U a_i, \sum_V a_i)R_a$. Hence the reverse implication holds for $q = n$ and the result follows by induction.

(2.6) Theorem. *Let R satisfy the intersection property. If* $(a_1, \cdots, a_n)R_a$, *then* $(\sum_U a_i)(\sum_V a_i) = \sum_{UV} a_i$ *for every* $U, V \subset [1, \cdots, n]$ *such that* $UV \neq \Theta$ and $j < k < m$ *for* $j \in U - UV$, $k \in V - UV$, $m \in UV$.

Proof. Let $W = U - UV$, $X = V - UV$. Then by the hypotheses,

$(\sum_X a_i, \ \sum_{UV} a_i)R_a$, $(\sum_W a_i, \ \sum_X a_i + \sum_{UV} a_i)R_a$, whence $(\sum_U a_i)(\sum_V a_i) = (\sum_W a_i + \sum_{UV} a_i)(\sum_X a_i + \sum_{UV} a_i) = \sum_{UV} a_i$ by virtue of the intersection property.

(2.7) LEMMA. *Let R satisfy the intersection property. If $(a_1, \cdots, a_n)R_a$, $U+V=[1, \cdots, n]$, $UV=\Theta$, then $(\sum_U a_i)(\sum_V a_i) = (a_n)(\sum_U a_i)(\sum_V a_i)$.*

PROOF. Let U, $V \neq \Theta$ and let $1 \in U$. Partition the set $[1, \cdots, n]$ with sets W_i defined so that $W_{2i-1} \subset U$, $W_{2i} \subset V$, and $j' < k'$ for $j' \in W_j$, $k' \in W_k$, $j<k$. (The existence of such a partition is readily proved inductively.) Then $1 \in W_1$ and for some m, $n \in W_m$. The result is immediate for $m=2$; let $m \geqq 3$. Define $b_j = \sum_{W_j} a_i$ for $1 \leqq j \leqq m$. Then $(\sum_U a_i)(\sum_V a_i) = (\sum_U a_i)(b_1 + \sum_3^m b_i)(b_2 + \sum_3^m b_i)(\sum_V a_i) = (\sum_U a_i)(\sum_3^m b_i)(\sum_V a_i)$, the last equality holding by virtue of the intersection property. For $m \geqq 4$, let $3 \leqq q < m$. Then

$$\left(\sum_1^{q-1} b_i + \sum_{q+1}^m b_i\right) \geqq \sum_U a_i$$

or $\sum_V a_i$ according as q is even or odd. Thus

$$\begin{aligned}\left(\sum_U a_i\right)\left(\sum_q^m b_i\right)\left(\sum_V a_i\right) &= \left(\sum_U a_i\right)\left(\sum_1^{q-1} b_i + \sum_{q+1}^m b_i\right)\left(b_q + \sum_{q+1}^m b_i\right)\left(\sum_V a_i\right)\\ &= \left(\sum_U a_i\right)\left(\sum_{q+1}^m b_i\right)\left(\sum_V a_i\right).\end{aligned}$$

Therefore

$$\left(\sum_U a_i\right)\left(\sum_V a_i\right) = \left(\sum_U a_i\right)(b_m)\left(\sum_V a_i\right).$$

Let $X = W_m - [n]$ with $X \neq \Theta$; otherwise, the proof is complete. Then $(\sum_1^{m-1} b_i + a_n) \geqq \sum_U a_i$ or $\sum_V a_i$ according as m is even or odd, whence

$$\begin{aligned}\left(\sum_U a_i\right)(b_m)\left(\sum_V a_i\right) &= \left(\sum_U a_i\right)\left(\sum_1^{m-1} b_i + a_n\right)\left(\sum_X a_i + a_n\right)\left(\sum_V a_i\right)\\ &= \left(\sum_U a_i\right)(a_n)\left(\sum_V a_i\right).\end{aligned}$$

Hence

$$\Big(\sum_U a_i\Big)\Big(\sum_V a_i\Big) = (a_n)\Big(\sum_U a_i\Big)\Big(\sum_V a_i\Big).$$

(2.8) THEOREM. *Let R satisfy the intersection property. If $(a_1, \cdots, a_n)R_a$, then for nonempty disjoint U, $V\subset[1, \cdots, n]$, $(\sum_U a_i)(\sum_V a_i)=a$.*

PROOF. The result is immediate for $n=2$. Suppose it holds for $q\leqq n-1$ where $n\geqq 3$. Then it holds for $U+V\neq[1, \cdots, n]$, with an application of (2.4.b). Let $U+V=[1, \cdots, n]$. From the lemma, $(\sum_U a_i)(\sum_V a_i)=(a_n)(\sum_U a_i)(\sum_V a_i)$. Let $n\in V$. Then for $V=[n]$, $(\sum_U a_i)(\sum_V a_i)=a$ by definition, and for $V\neq[n]$, $(\sum_U a_i)(\sum_V a_i)$ $=(a_n)(\sum_U a_i)(\sum_V a_i)=a\sum_V a_i=a$ by the induction hypothesis. Similarly, for $n\in U$, $(\sum_U a_i)(\sum_V a_i)=a$. Hence the result holds for $n=q$ and the proof is complete.

(2.9) DEFINITION. Define $(a_1, \cdots, a_n)\overline{R}_a$ (read $(a_1, \cdots, a_n)$ *symmetrically R-independent over a*) to mean $(a_{i_1}, \cdots, a_{i_n})R_a$ for every permutation $(i_1, \cdots, i_n)$ of the integers $[1, \cdots, n]$.

(2.10) COROLLARY. (a) *The relation $\overline{R}_a$ is symmetric.* (b) *If $(a_1, \cdots, a_n)\overline{R}_a$, then $(a_1, \cdots, a_n)R_a$.*

(2.11) THEOREM. *If $(a_1, \cdots, a_n)\overline{R}_a$, then $(a_j, \sum_{i\neq j} a_i)\overline{R}_a$ for $1\leqq j\leqq n$, and conversely, provided $(a_j, \sum_V a_i)T$ for every nonempty $V\subset[1, \cdots, n]$ such that $j\notin V$.*

PROOF. This follows from (2.5) in a manner similar to the corresponding result in [**5**].

(2.12) THEOREM. *Let R satisfy the intersection property. If $(a_1, \cdots, a_n)\overline{R}_a$, then $(\sum_U a_i)(\sum_V a_i)=\sum_{UV} a_i$ for every U, V $\subset[1, \cdots, n]$ such that $UV\neq\Theta$.*

PROOF. Let $U\not\subset V$ and $V\not\subset U$. Then let $U-UV=[i_1, \cdots, i_u]$, $V-UV=[j_1, \cdots, j_v]$, $UV=[k_1, \cdots, k_w]$, where the i_m, j_m and k_m are distinct. Define

$$b_m = \begin{cases} a_{i_m} & \text{for } 1 \leqq m \leqq u, \\ a_{j_{m-u}} & \text{for } u+1 \leqq m \leqq u+v, \\ a_{k_{m-u-v}} & \text{for } u+v+1 \leqq m \leqq u+v+w. \end{cases}$$

Then $(b_1, \cdots, b_{u+v+w})R_a$ by (2.9) and (2.4.b). Also

$$U' = [1, \cdots, u, u+v+1, \cdots, u+v+w]$$

and

$$V' = [u + 1, \cdots, u + v + w]$$

satisfy the hypotheses of (2.6), whence

$$\left(\sum_U a_i\right)\left(\sum_V a_i\right) = \left(\sum_{U'} b_i\right)\left(\sum_{V'} b_i\right) = \sum_{U'V'} b_i = \sum_{UV} a_i.$$

In the remainder of this section, some results are stated for R symmetric at a. The proofs of these results are similar to those of the corresponding results in [5] and will be omitted. In case R were a symmetric relation, it is evident that R would be symmetric at a for every $a \in L$. If R is symmetric at a, then the relation R_a is symmetric, or equivalently, $(b, c)R_a$ if and only if $(b, c)\overline{R}_a$.

(2.13) LEMMA. *Let R be symmetric at a. If $(c, b, d)R_a$, then $(b, c, d)R_a$.*

(2.14) THEOREM. *If R is symmetric at a, then $(a_1, \cdots, a_n)\overline{R}_a$ if and only if $(a_1, \cdots, a_n)R_a$.*

(2.15) COROLLARY. *If R is symmetric at a, then $(a_1, \cdots, a_n)R_a$ if and only if $(\sum_U a_i, \sum_V a_i)R_a$ for every nonempty disjoint $U, V \subset [1, \cdots, n]$.*

(2.16) THEOREM. *Let R be symmetric at a and let $b_1, \cdots, b_m \in L$ where $m \geqq 2$. If $(a_1, \cdots, a_n)R_a$, $(b_1, \cdots, b_m)R_a$ and $(\sum_1^n a_i, \sum_1^m b_i)R_a$, then $(a_1, \cdots, a_n, b_1, \cdots, b_m)R_a$.*

(2.17) COROLLARY. *Let R be symmetric at a and for $j = 1, \cdots, n$, let $m_j \geqq 2$ and $a_{ij} \in L$ for $i = 1, \cdots, m_j$. If $(a_{1j}, \cdots, a_{m_j j})R_a$ for $j = 1, \cdots, n$ and if $(\sum_1^{m_1} a_{i1}, \cdots, \sum_1^{m_n} a_{in})R_a$, then*

$$(a_{11}, \cdots, a_{m_1 1}, \cdots, a_{1n}, \cdots, a_{m_n n})R_a.$$

3. Quasi-modularity relations. In the study of quasi-dual-ideals, the relations of weak modularity, as denoted by Wilcox [7], and quasi-modularity, as denoted by the author [3], arise with properties similar to those of ordinary modularity. In this section the material of §2 is applied in an abstraction of these relations.

(3.1) DEFINITION. A nonempty subset S of L is a *quasi-dual-ideal* (q.d.i.) if

(a) $x \in S$, $y \geqq x$ implies $y \in S$;

(b) $x, y \in S$, $(x, y)M$ implies $xy \in S$.

The smallest q.d.i. containing a set T (or elements $a, b, c, \cdots$) is denoted by $\{T\}$ (or $\{a, b, c, \cdots\}$). The set of all q.d.i. is $\mathfrak{L}$ and the set of all principal q.d.i. (of the form $\{a\}$) is $\mathfrak{S}$. For $\alpha, \beta \in \mathfrak{L}$,

$\alpha \leqq \beta$ means $\alpha \supset \beta$, $\alpha \cup \beta = \alpha \cdot \beta$ and $\alpha \cap \beta = \{\alpha + \beta\}$.

It is useful to note that the principal q.d.i. of L coincide with the principal dual ideals of L. For the next corollary and for all statements with reference to $\mathfrak{L}$ in the remainder of the paper, it is assumed that l.u.b. $L = 1$ exists.

(3.2) Corollary. *The set $\mathfrak{L}$ is a complete lattice with respect to $\leqq$; the lattice operations are $\cup$, $\cap$, and L and $\{1\}$ are the zero and unit respectively. If $(b, c)M$, $\{b, c\} = \{bc\}$. The lattice L is isomorphic to the set $\mathfrak{S}$, a lattice subset (not necessarily a sublattice) of $\mathfrak{L}$, under $a \to \{a\}$.*

Proof. In $\mathfrak{S}$, l.u.b. $[\{a\}, \{b\}] = \{a+b\}$ and g.l.b. $[\{a\}, \{b\}] = \{ab\}$. The isomorphism now follows and the remainder is immediate.

(3.3) Definition. Let $Q \subset L \times L$. Then Q is a *quasi-modularity relation* means that $\mathfrak{Q} = [(\{b\}, \{c\}); (b, c)Q]$ is a modularity relation under $\mathfrak{S} \times \mathfrak{S}$ in $\mathfrak{L}$. For Q a quasi-modularity relation, Q is said to satisfy the *intersection property* (to be *symmetric at* α, for $\alpha \in \mathfrak{L}$) if $\mathfrak{Q}$ satisfies the intersection property (if $\mathfrak{Q}$ is symmetric at α) in $\mathfrak{L}$.

(3.4) Definition. Let Q be a quasi-modularity relation. For $n \geqq 2$, $a_1, \cdots, a_n \in L$ and $\alpha \in \mathfrak{L}$, $(a_1, \cdots, a_n)Q_\alpha$ (read $(a_1, \cdots, a_n)$ *Q-quasi-independent over* α) means $(\{a_1\}, \cdots, \{a_n\})\mathfrak{Q}_\alpha$ where $\mathfrak{Q}$ is defined as in (3.3).

(3.5) Corollary. *If $(a_1, \cdots, a_n)Q_\alpha$, then $(\sum_U a_i, \sum_V a_i)Q$, $\{\sum_U a_i, \sum_V a_i\} = \alpha$ for every nonempty $U, V \subset [1, \cdots, n]$ such that $j < k$ for $j \in U$, $k \in V$, and conversely.*

The corollary shows the analogy between Q-quasi-independence over a q.d.i. of L and R-independence over an element of L as defined in (2.3). The results of the independence theory of the previous section may be applied to $\mathfrak{Q}$, yielding a corresponding theory for Q. If one keeps in mind the equalities $\{b\} \cup \{c\} = \{b+c\}$, $\{b\} \cap \{c\} = \{b, c\}$ and that $\alpha \leqq \{a\}$ means $a \in \alpha$, the independence theory for Q may be stated free of the notation of the lattice $\mathfrak{L}$.

4. **Examples.** An example of a modularity relation is obtained from a special case of relative modularity, the latter being a relativization of ordinary modularity.

(4.1) Definition. For $S \subset L$, $b, c \in L$, $(b, c)M_S$ (read (b, c) *modular relative to* S) means $(a+b)c = a+bc$ for every $a \in S$ such that $a \leqq c$.

Evidently, $M = M_L$. In addition, M_S satisfies many of the properties of M, some in a modified form. In particular, the next lemma is of interest.

(4.2) LEMMA. *If* $(b, c)M_S$, $b' \leqq b$, $c' \leqq c$, $b'c' = bc$, *then* $(b', c')M_S$.

PROOF. Let $a \leqq c'$, $a \in S$. Then $(a+b')c' \leqq (a+b)c = a+bc = a+b'c'$, whence $(b', c')M_S$ since the reverse inequality $(a+b')c' \geqq a+b'c'$ holds universally for $a \leqq c'$.

(4.3) THEOREM. *If S is join-closed, then* $R = (S \times L) \cdot M_S$ *is a modularity relation under* $S \times L$.

PROOF. Part (a) of (2.1) readily follows with an application of (4.2). For Part (b), let $(c, d)R$, $(b, c+d)R$, $b(c+d) = cd$. Then $b, c \in S$, $(b+c, d) \in S \times L$ and $b(c+d) \leqq c$. Now let $a \leqq d$, $a \in S$. Then $a+c \in S$, $a+c \leqq c+d$ and

$$\begin{aligned}(a + (b + c))d &= ((a + c) + b)(c + d)d = ((a + c) + b(c + d))d \\ &= (a + (c + b(c + d)))d = (a + c)d \\ &= a + cd \leqq a + (b + c)d.\end{aligned}$$

Thus $(b+c, d)M_S$, whence $(b+c, d)R$. Also

$$(b + c)d = (c + b)(c + d)d = (c + b(c + d))d = cd.$$

(4.4) THEOREM. *If S is join-closed, then* $R = (S \times S) \cdot M_S$ *is a modularity relation under* $S \times S$ *satisfying the intersection property.*

PROOF. The proof that R is a modularity relation under $S \times S$ is essentially the proof of (4.3). For the remainder, let $(c, d)R$, $(b, c+d)R$, $b(c+d) = cd$. Then $d \in S$, $b(c+d) \leqq d$ and since $(b, c+d)M_S$, $(b+d) \cdot (c+d) = d + b(c+d) = d$.

Two examples of quasi-modularity relations are now considered.

(4.5) DEFINITION. For $b, c \in L$,

(a) $(b, c)M_0$ (read (b, c) *weakly modular*) means $\{a+b, c\} = \{a\} \cup \{b, c\}$ for every $a \leqq c$;

(b) $(b, c)M_q$ (read (b, c) *quasi-modular*) means $(b, c)M_S$ where $S = \{b, c\}$.

(4.6) THEOREM. *The relations M_0 and M_q are quasi-modularity relations satisfying the intersection property.*

The proof of this theorem is omitted. It is of interest to note that always $M_0 \subset M_q$ and that examples of left-complemented [6] lattices exist for which the inclusion is proper.

To show that the notion of a modularity relation is more general than ordinary modularity, one may consider the relation $\mathfrak{Q}$ in $\mathfrak{L}$ corresponding to M_0, which is incidentally $(\mathfrak{S} \times \mathfrak{S}) \cdot M_{\mathfrak{S}}$. In case L is not a modular lattice, this $\mathfrak{Q}$, although a modularity relation, is not ordinary modularity for $\mathfrak{L}$.

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